Smooth Shape-Aware Functions with Controlled Extrema

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Real-time deformation relies on smooth, shape-aware functions

input shape + handles
Real-time deformation relies on smooth, shape-aware functions

precompute weight functions
Real-time deformation relies on smooth, shape-aware functions

deform handles → deform shape
Real-time deformation relies on smooth, shape-aware functions

\[ x'_i = \sum_{j=1}^{H} f_j(x_i) T_j x_i \]
Real-time deformation relies on smooth, shape-aware functions

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Spurious extrema cause distracting artifacts

unconstrained $\Delta^2$

[Botsch & Kobbelt 2004]

$$x'_i = \sum_{j=1}^{H} f_j(x_i) T_j x_i$$

local max  
local min
Spurious extrema cause distracting artifacts

unconstrained $\Delta^2$
[Botsch & Kobbelt 2004]

$$x'_i = \sum_{j=1}^{H} f_j(x_i) T_j x_i$$
Bounds help, but don’t solve problem

bounded $\Delta^2$

[Jacobson et al. 2011]

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$$x'_i = \sum_{j=1}^{H} f_j(x_i) T_j x_i$$
Gets worse with higher-order smoothness

bounded $\Delta^4$

[Jacobson et al. 2011]

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$\Delta^k, k > 2$ oscillate too much
Gets worse with higher-order smoothness

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[Jacobson et al. 2011]

$$x'_i = \sum_{j=1}^{H} f_j(x_i) T_j x_i$$

$\Delta^k, k > 2$ oscillate too much
We explicitly prohibit spurious extrema

$$x'_i = \sum_{j=1}^{H} f_j(x_i) T_j x_i$$

our $\Delta^4$

<table>
<thead>
<tr>
<th>local max</th>
<th>local min</th>
</tr>
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</table>

August 9, 2012    Alec Jacobson    # 13
We explicitly prohibit spurious extrema

\[ x'_i = \sum_{j=1}^{H} f_j(x_i) T_j x_i \]
Same functions used for color interpolation

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\[ c_i = \sum_{j=1}^{H} f_j(x_i) c_j \]
Same functions used for color interpolation

unconstrained $\Delta^2$

[Finch et al. 2011]

$$c_i = \sum_{j=1}^{H} f_j(x_i)c_j$$

Image courtesy Mark Finch
Same functions used for color interpolation

unconstrained $\Delta^2$
[Finch et al. 2011]

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$$c_i = \sum_{j=1}^{H} f_j(x_i)c_j$$

Our $\Delta^2$
Want same control when smoothing data
Want same control when smoothing data

Exact, but sharp geodesic
Want same control when smoothing data

Exact, but sharp geodesic
Want same control when smoothing data

Exact, but sharp geodesic
Smooth, but extrema are lost
Want same control when smoothing data

Exact, but sharp geodesic

Smooth and maintain extrema
Ideal discrete problem is intractable

$$\arg \min_f E(f)$$

Interpolation functions:

$$E_L(f) = \int_{\mathcal{M}} \| \nabla^k f \|^2 dV, \quad k = 2, 3, \ldots$$
Ideal discrete problem is intractable

$$\arg\min \ E(f)$$

Data smoothing:

$$E_L(f) = \int_M \| \nabla^k f \|^2 dV, \quad k = 2, 3, \ldots$$

$$E_D(f) = \sum_{i \in M} \| h_i - f_i \|^2$$

$$E(f) = \gamma_L E_L(f) + \gamma_D E_D(f)$$
Ideal discrete problem is intractable

$$\arg\min_{f} E(f)$$
Ideal discrete problem is intractable

$$\arg \min_f \ E(f)$$

s.t.  \( f_{\text{max}} = \text{known} \)

\( f_{\text{min}} = \text{known} \)
Ideal discrete problem is intractable

\[ \arg \min_{f} E(f) \]

s.t. \[ f_{\text{max}} = \text{known} \]
\[ f_{\text{min}} = \text{known} \]

linear \[ f_j < f_{\text{max}} \]
\[ f_j > f_{\text{min}} \]
## Ideal discrete problem is intractable

\[
\arg \min_f \ E(f)
\]

**s.t.** 
\[
\begin{align*}
    f_{\text{max}} &= \text{known} \\
    f_{\text{min}} &= \text{known}
\end{align*}
\]

**linear**
\[
\begin{align*}
    f_j &< f_{\text{max}} \\
    f_j &> f_{\text{min}}
\end{align*}
\]

**nonlinear**
\[
\begin{align*}
    f_i &> \min_{j \in \mathcal{N}(i)} f_j \\
    f_i &< \max_{j \in \mathcal{N}(i)} f_j
\end{align*}
\]
Assume we have a feasible solution

\[
\arg\min_{f} E(f)
\]

s.t. \[f_{\text{max}} = \text{known}\]
\[f_{\text{min}} = \text{known}\]

**linear**
- \[f_j < f_{\text{max}}\]
- \[f_j > f_{\text{min}}\]

**nonlinear**
- \[f_i > \min_{j \in \mathcal{N}(i)} f_j\]
- \[f_i < \max_{j \in \mathcal{N}(i)} f_j\]

**“Representative function”** \( U \)

- \[u_j < u_{\text{max}}\]
- \[u_j > u_{\text{min}}\]

\( u_i > \min_{j \in \mathcal{N}(i)} u_j \)
\( u_i < \max_{j \in \mathcal{N}(i)} u_j \)
Assume we have a feasible solution

“Representative function” $\mathcal{U}$

<table>
<thead>
<tr>
<th>handles</th>
<th>$u_j &lt; u_{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_j &gt; u_{\min}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>interior</th>
<th>$u_i &gt; \min_{j \in \mathcal{N}(i)} u_j$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$u_i &lt; \max_{j \in \mathcal{N}(i)} u_j$</td>
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Copy “monotonicity” of representative

\[ \arg \min_f E(f) \]

s.t. \( f_{\max} = \text{known} \)

\( f_{\min} = \text{known} \)

\[ (f_i - f_j)(u_i - u_j) > 0 \quad \text{linear} \quad \forall (i, j) \in \mathcal{E} \]

At least one edge in either direction per vertex
Rewrite as conic optimization

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| F f \|^2 + c^T f + \text{const} \\
\text{subject to} & \quad A^T_{\text{eq}} f \leq b_{\text{eq}}, \\
& \quad f \leq u_f, \quad f \geq l_f
\end{align*}
\]
We always have harmonic representative

$$\arg \min_u \frac{1}{2} \int_{\Omega} \| \nabla u \|^2 dV$$
We always have harmonic representative

$$\arg\min_u \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 dV$$

s.t. \hspace{1cm} u_{\text{max}} = 1
We always have harmonic representative

\[
\arg\min_u \frac{1}{2} \int_{\Omega} \| \nabla u \|^2 dV
\]

s.t. \quad u_{\text{max}} = 1

s.t. \quad u_{\text{min}} = 0
We always have harmonic representative

$$\arg\min_u \frac{1}{2} \int_{\Omega} \| \nabla u \|^2 dV$$

s.t. \( u_{\max} = 1 \)

s.t. \( u_{\min} = 0 \)

Works well when no input function exists
Data energy may fight harmonic representative

Anisotropic input data
Data energy may fight harmonic representative

Anisotropic input data

Harmonic representative
Data energy may fight harmonic representative
Data energy may fight harmonic representative

Anisotropic input data

Harmonic representative
Data energy may fight harmonic representative

Anisotropic input data

Resulting solution with large $\gamma_D$
If data exists, copy topology, too

Anisotropic input data

[Weinkauf et al. 2010] representative
If data exists, copy topology, too

Anisotropic input data

Resulting solution with large $\gamma D$
Final algorithm is simple and efficient

- *Data smoothing*: topology-aware representative
  - Morse-smale + linear solve ~milliseconds
Final algorithm is simple and efficient

- **Data smoothing**: topology-aware representative
  - Morse-smale + linear solve ~milliseconds
- **Interpolation**: harmonic representative
  - Linear solve ~milliseconds
Final algorithm is simple and efficient

- **Data smoothing**: topology-aware representative
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- **Interpolation**: harmonic representative
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- **Conic optimization**
  - 2D ~milliseconds, 3D ~seconds
Final algorithm is simple and efficient

- **Data smoothing**: topology-aware representative
  - Morse-smale + linear solve ~milliseconds

- **Interpolation**: harmonic representative
  - Linear solve ~milliseconds

- **Conic optimization**
  - 2D ~milliseconds, 3D ~seconds

**Interpolation**: functions are precomputed
We preserve troublesome appendages

Bounded $\Delta^2$

Our $\Delta^2$
We preserve troublesome appendages

Bounded $\Delta^2$

Our $\Delta^2$
We preserve troublesome appendages

Bounded $\Delta^2$

Our $\Delta^2$
Our weights attach appendages to body

[Botsch & Kobbelt 2004, Jacobson et al. 2011]
Extrema glue appendages to far-away handles

[Botsch & Kobbelt 2004, Jacobson et al. 2011]
Extrema glue appendages to far-away handles

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Our weights attach appendages to body

Our method
Our weights attach appendages to body

Our method
Extrema distort small features

Unconstrained $\Delta^2$ [Botsch & Kobbelt 2004]

weight of middle point
Extrema distort small features

Unconstrained $\Delta^2$ [Botsch & Kobbelt 2004]

weight of middle point
Extrema distort small features

Bounded $\Delta^2$ [Jacobson et al. 2011]

weight of middle point
“Monotonicity” helps preserve small features

Bounded $\Delta^2$ [Jacobson et al. 2011]

Our $\Delta^2$
Spurious extrema are unstable, may “flip”
Spurious extrema are unstable, may “flip”
Spurious extrema are unstable, may “flip”

Unconstrained $\Delta^3$ [Botsch & Kobbelt, 2004]
Spurious extrema are unstable, may “flip”

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Unconstrained $\Delta^3$ [Botsch & Kobbelt, 2004]
Spurious extrema are unstable, may “flip.”
Spurious extrema are unstable, may “flip”

Bounded $\Delta^3$
Lack of extrema leads to more stability

Our $\Delta^3$
Lack of extrema leads to more stability

Our $\Delta^3$
Even control continuity at extrema

Our method without data term

Our method with data term
Even control continuity at extrema

Original

Direct extension of [Botsch & Kobbelt 2004]
Even control continuity at extrema

Original

[Botsch & Kobbelt 2004] + data term
Even control continuity at extrema

Original

Our method without data term
Even control continuity at extrema

Original

Our method with data term
Reproduces results of Weinkauf et al. 2010...
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Original noisy data
Reproduces results of Weinkauf et al. 2010...

Original noisy data

Simplified and smoothed
Reproduces results of Weinkauf et al. 2010...

Original noisy data

Simplified and smoothed
… but 1000 times faster

30K vertices
5 seconds per solve
... but 1000 times faster

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Conclusion: Important to control extrema

- Copy “monotonicity” of harmonic functions
- *Reduces* search-space, but optimization is tractable
Future work and discussion

- Larger, but still tractable subspace?
  - Consider all valid harmonic functions?
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- Larger, but still tractable subspace?
  - Consider all valid harmonic functions?

- Continuous formulation?
Acknowledgements

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MATLAB Demo:
http://igl.ethz.ch/projects/monotonic/

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