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Topology-based Smoothing of 2D Scalar Fields with C^1-Continuity: Derivatives of f

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We provide the first and second derivatives of the function value f_i for a free vertex i of the monotonicity graph (any node with a single incoming edge). Let $parent(i)$ be the parent of vertex i in the monotonicity graph, and let f_{m_i} be the maximum function value at the end of all paths in the graph passing through vertex i. Recall that we substitute the variables f_i at free vertices by using new variables θ_i, where

$$f_i = f_i(\theta_i) = f_{m_i} + t_i(\theta_i)\left(f_{parent(i)} - f_{m_i}\right),$$

such that

$$t_i = 0.5 + 0.5\cos(\theta_i).$$

For convenience, define $parent^n(i)$ to be the n-th ancestor of vertex i:

$$\begin{aligned} parent^0(i) &= i \\ parent^1(i) &= parent(i) \\ parent^2(i) &= parent(parent(i)) \\ &\vdots \end{aligned}$$

Then the first derivative of f_i with respect to θ_j, where vertex j is the p-th ancestor of vertex i, is

$$\frac{\partial f_i}{\partial \theta_j} = \left(\prod_{a=0}^{a<p} t_{parent^a(i)}\right)\frac{-\sin(\theta_j)}{2}\left(f_{parent(j)} - f_{m_j}\right).$$

The derivative is 0 if vertex j is not an ancestor of vertex i or is i itself.

The second derivative of f_i with respect to θ_j and θ_k follows. Let vertex j be the p-th ancestor of vertex i and vertex k be the q-th ancestor of vertex i (p and/or q may be 0). Without loss of generality, assume $p < q$.
If $j \neq k$,

$$\frac{\partial f_i}{\partial \theta_j \partial \theta_k} = \left(\prod_{a=0}^{a<p} t_{parent^a(i)}\right)\frac{-\sin(\theta_j)}{2}\left(\prod_{a=p+1}^{a<q} t_{parent^a(i)}\right)\frac{-\sin(\theta_k)}{2}\left(f_{parent(k)} - f_{m_k}\right).$$

If $j = k$,

$$\frac{\partial^2 f_i}{\partial \theta_j^2} = \left(\prod_{a=0}^{a<p} t_{parent^a(i)}\right)\frac{-\cos(\theta_j)}{2}\left(f_{parent(j)} - f_{m_j}\right).$$

If even one of vertex j or vertex k is not an ancestor of vertex i (or vertex i itself),

$$\frac{\partial^2 f_i}{\partial \theta_j \partial \theta_k} = 0.$$

Recall our energy functional

$$E(f) = \int_\Omega |\Delta f|^2 + \omega_d \left|f - \hat{f}\right|^2 dA,$$

where Ω is our parametric domain, $\hat{f}$ is the original function and $\omega_d > 0$ is the weight of the data term. We discretize this as

$$E(f) = \sum_i^N (\Delta f_i)^2 + \omega_d \left(f_i - \hat{f}_i\right)^2,$$

where Δf_i is the discrete Laplace operator value integrated over the area cell around vertex i. Then the gradient of E is

$$\nabla_\theta E(f) = 2 \sum_i^N (\Delta f_i)\, \nabla_\theta \Delta f_i + \omega_d \left(f_i - \hat{f}_i\right) \nabla_\theta f_i$$

and the Hessian matrix of second partial derivatives is

$$H_\theta E(f) = 2 \sum_i^N (\Delta f_i)\, H_\theta \Delta f_i + (\nabla_\theta \Delta f_i)^T (\nabla_\theta \Delta f_i) + \omega_d \left[\left(f_i - \hat{f}_i\right) H_\theta f_i + (\nabla_\theta f_i)^T (\nabla_\theta f_i)\right].$$

The discretization of the Laplace operator,

$$\Delta f_i = \sum_{j:(i,j)\in\mathcal{E}} 0.5 \left(\cot\alpha_{ij} + \cot\beta_{ij}\right)(f_i - f_j),$$

has as its gradient

$$\nabla_\theta \Delta f_i = \sum_{j:(i,j)\in\mathcal{E}} 0.5 \left(\cot\alpha_{ij} + \cot\beta_{ij}\right)(\nabla_\theta f_i - \nabla_\theta f_j)$$

and as its Hessian

$$H_\theta \Delta f_i = \sum_{j:(i,j)\in\mathcal{E}} 0.5 \left(\cot\alpha_{ij} + \cot\beta_{ij}\right)(H_\theta f_i - H_\theta f_j).$$

Finally, $\nabla_\theta f_i$ and $H_\theta f_i$ are composed of the $\frac{\partial f_i}{\partial \theta_j}$ and $\frac{\partial^2 f_i}{\partial \theta_j \partial \theta_k}$ expressions, given above.