

Why Linear Algebra?

(for Computer Scientists)

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Why Math?

- Computer Scientists,
not just programmers
- Practice solid arguments,
correctness proofs -
it's an art!



Main topics of the LA class

- Linear systems of equations

$$3x_1 + 4x_2 - 1.5x_3 = 0$$

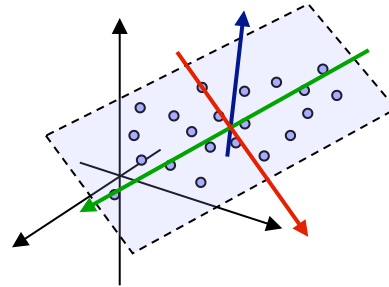
$$x_1 - 3.2x_2 + 5x_3 = 17$$

$$2x_1 + 7x_2 + 3.1x_3 = 42$$

$$A\mathbf{x} = \mathbf{b}$$

- Linear (vector) spaces and transformations

$$\mathbf{x}, \mathbf{y} \in V \Rightarrow \alpha\mathbf{x} + \beta\mathbf{y} \in V$$



Linear Algebra is everywhere

- Most world's phenomena involve complicated equations
- Computers can only do basic arithmetic
- → Usually can't do the original equations, approximate by series of **linear equations**
- → Model things as **linear spaces**

$$E(q) = \sum_{nT} A_T \|\nabla q^T - \mathbf{w}^T\|^2 \rightarrow \min$$

$$E(V') = \sum_{i=1} \|\delta_i - \mathcal{L}(\mathbf{v}'_i)\|^2 + \sum_{i=m}^n \|\mathbf{v}'_i - \mathbf{u}_i\|^2,$$

$$q_m^i = \sum_k \frac{1}{M_i - 1} \sum_{j \in O^k} X^{ji} \sum_{n=1}^3 w_{mn}^{ij} q_n^j$$

$$\arg \min_{w_j} \frac{1}{2} \int_{\Omega} |\Delta w_j|^2 dV$$

$$E(q) = \sum_T A_T \|\nabla q^T - \mathbf{w}^T\|^2 \rightarrow \min$$

$$\tilde{I}^i(\cdot) : \bigcup_{k=1}^{d_i-1} \Delta_k^i \rightarrow \mathbb{R}.$$

$$\tilde{I}^i(\mu) = \langle \mu, \mu \rangle_{\mathbb{R}^3} = \langle \mu_1 \tilde{\mathbf{x}}_k^i + \mu_2 \tilde{\mathbf{x}}_{k+1}^i, \mu_1 \tilde{\mathbf{x}}_k^i + \mu_2 \tilde{\mathbf{x}}_{k+1}^i \rangle_{\mathbb{R}^3} =$$

$$= \mu_1^2 \tilde{g}_{k,k}^i + 2 \mu_1 \mu_2 \tilde{g}_{k,k+1}^i + \mu_2^2 \tilde{g}_{k+1,k+1}^i,$$

$$\hat{\mathbf{x}}_2^i = \langle \tilde{\mathbf{x}}_2^i, \frac{\tilde{\mathbf{x}}_1^i}{\|\tilde{\mathbf{x}}_1^i\|} \rangle \frac{\tilde{\mathbf{x}}_1^i}{\|\tilde{\mathbf{x}}_1^i\|} + \langle \tilde{\mathbf{x}}_2^i, \mathbf{n} \rangle \mathbf{n} + \langle \tilde{\mathbf{x}}_2^i, \mathbf{N}^i \rangle \mathbf{N}^i =$$

Impossible without Computation and Linear Algebra

Examples from everyday life

Weather forecasting

- Solve PDEs (partial differential equations) that model the physics of the atmosphere
- Unknowns: temperature, humidity, wind... at every point in Earth's atmosphere at a certain time

Temperature

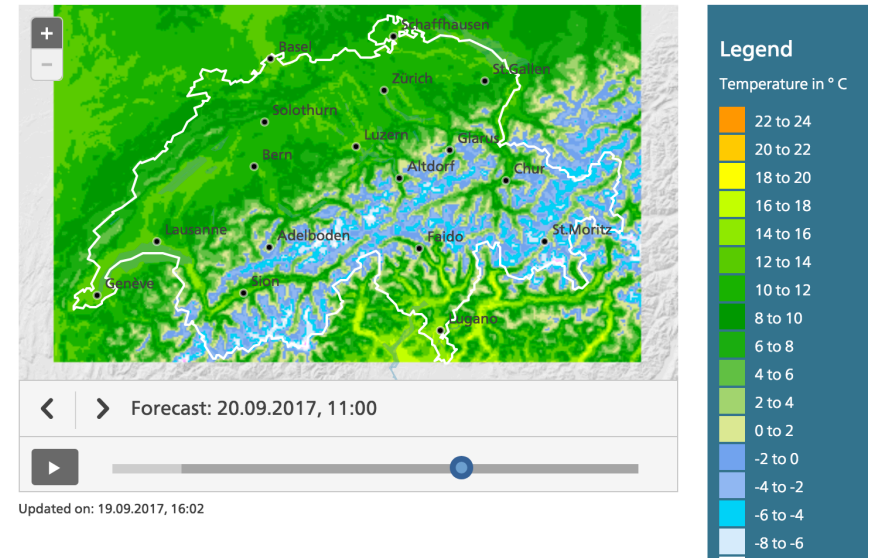


Image source: MeteoSwiss

Weather forecasting

- Analytical solution (formula) doesn't exist
- → Discretization on a grid, numerical approximation
- Huge systems of linear equations

$$Ax = b$$

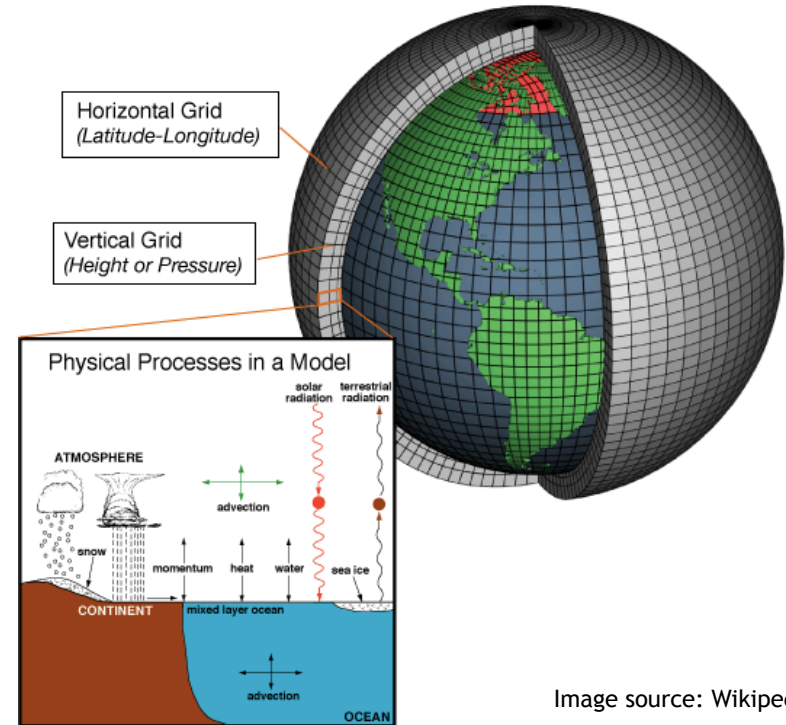


Image source: Wikipedia

Weather forecasting

- Linear algebra done by supercomputers!
- CS challenge: how to solve huge linear equations, and fast



Some of the MeteoSwiss supercomputers at CSCS, Lugano

Google search engine

- Web crawler “reads” the Internet pages and indexes by keywords
- User enters keyword, search engine retrieves pages containing it
- In what order to present the found pages??



linear algebra



Google Search

I'm Feeling Lucky

[Linear algebra | Khan Academy](https://www.khanacademy.org/math/linear-algebra)

<https://www.khanacademy.org/math/linear-algebra> ▾ Khan Academy ▾

For anyone looking to review/preview **linear algebra**, this sequence of videos by one of our content creators Grant Sanderson illustrates many of the geometric ...

[Matrix transformations](#) · [Alternate coordinate systems](#) · [Integral calculus](#) · [Vectors](#)

[Linear algebra - Wikipedia, the free encyclopedia](https://en.wikipedia.org/wiki/Linear_algebra)

https://en.wikipedia.org/wiki/Linear_algebra ▾ Wikipedia ▾

Linear algebra is the branch of mathematics concerning vector spaces and linear mappings between such spaces. It includes the study of lines, planes, and subspaces, but is also concerned with properties common to all vector spaces.

[Rank](#) · [Basis](#) · [Kernel](#) · [Projection](#)

^[PDF] [Linear Algebra - UC Davis Mathematics](https://www.math.ucdavis.edu/~linear/linear-guest.pdf)

<https://www.math.ucdavis.edu/~linear/linear-guest.pdf> ▾ University of California, Davis ▾

Linear algebra is the study of vectors and linear functions. In broad terms, vectors are The main lesson of an introductory **linear algebra** course is this: you.

Google search engine - PageRank

- PageRank algorithm sorts search results by importance
- Importance of a page = how many other important pages link to it

$$\text{PageRank}(u) = \sum_{v: v \text{ links to } u} \frac{\text{PageRank}(v)}{\# \text{ links from } v}$$

- PageRanks of all webpages?
Eigenvalue problem! $A\mathbf{u} = \lambda\mathbf{u}$
- We will learn about it in the 2nd half of the semester

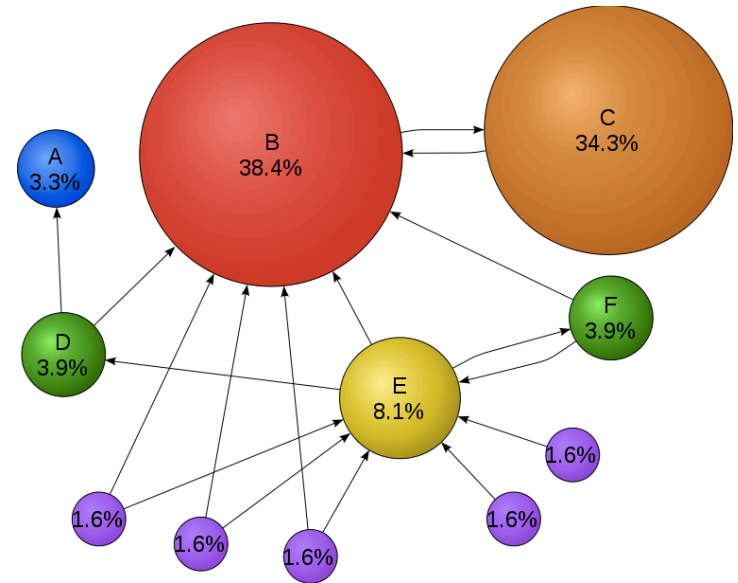


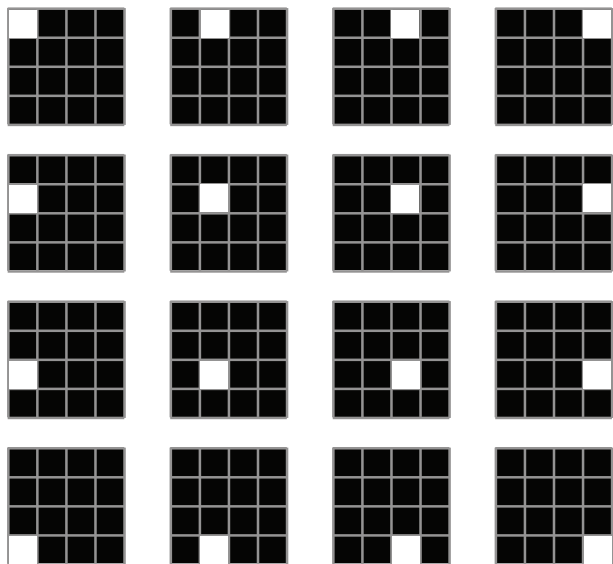
Image source: Wikipedia

Digital image representation

- Images are vectors!
- The image on the right:
 - 2272 x 1704 pixels
 - pixel = (R,G,B)-value
 - this image is a 11,614,464-dimensional vector



Images as vectors



The standard basis for 4x4 grayscale images
16 vectors

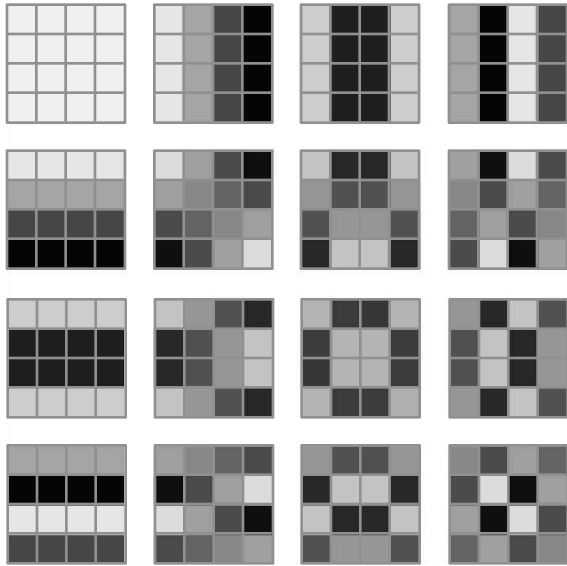
Any 4x4 grayscale image is a
linear combination of this standard basis

$$\text{Image} = 1 * \mathbf{b}_1 + 0.6 * \mathbf{b}_2 + \dots$$

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \dots + \alpha_n \mathbf{b}_n$$

Need to store all $\alpha_1, \alpha_2, \dots, \alpha_n$

JPEG image compression



The 4x4 DCT (discrete cosine) basis
16 vectors

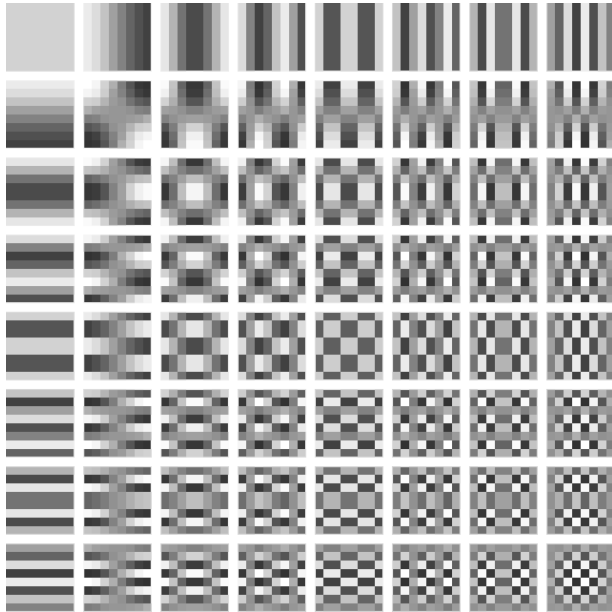
Any 4x4 grayscale image is also a
linear combination of that basis!

$$\text{Image} = 1.1 * \text{Basis}_1 + 0.9 * \text{Basis}_2 + \dots$$

$$\mathbf{x} = \beta_1 \mathbf{c}_1 + \beta_2 \mathbf{c}_2 + \dots + \beta_n \mathbf{c}_n$$

For “natural” images we can omit
all but a few first β

JPEG image compression



The 8x8 DCT (discrete cosine) basis
64 vectors

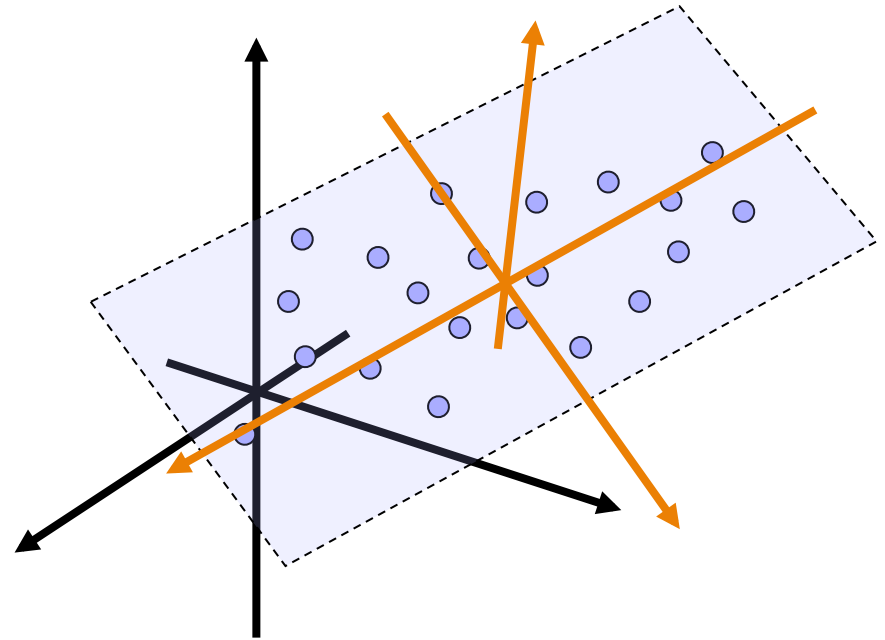
Any 8x8 grayscale image is a
linear combination of that basis!

$$\mathbf{x} = \beta_1 \mathbf{c}_1 + \beta_2 \mathbf{c}_2 + \dots + \beta_n \mathbf{c}_n$$

For “natural” images we can omit
all but a few first β

JPEG image compression

- Images are vectors in a (high-dimensional) space
- Different coordinate systems = different bases
- JPEG image compression: project onto a lower-dimensional linear space



Computer animation

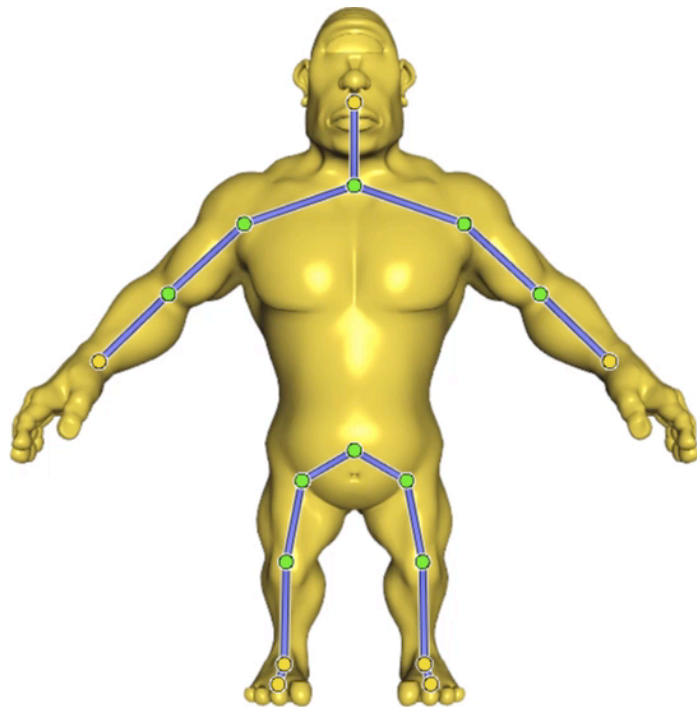
- How do virtual characters move?



Excerpt from “Big Buck Bunny”, open Blender movie

Computer animation

- Artist designs key poses for skeleton
- Collection of linear transformations in 3D space
- Automatic interpolation over the character's surface and over time



Linear Algebra is fundamental

Enjoy the class!

Interactive sessions: Clicker

- Install the **ETH EduApp** on your smartphone
 - or -
- Make sure you can log in at <https://eduapp-app1.ethz.ch/>

