Why Linear Algebra?

(for Computer Scientists)

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Why Math?

- Computer Scientists, not just programmers
- Practice solid arguments, correctness proofs – it's an art!

Main topics of the LA class

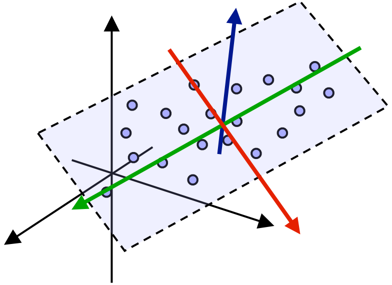
- Linear systems of equations

$$\begin{aligned} 3x_1 + 4x_2 - 1.5x_3 &= 0 \\ x_1 - 3.2x_2 + 5x_3 &= 17 \\ 2x_1 + 7x_2 + 3.1x_3 &= 42 \end{aligned}$$

$$A\mathbf{x} = \mathbf{b}$$

- Linear (vector) spaces and transformations

$$\mathbf{x}, \mathbf{y} \in V \Rightarrow \alpha\mathbf{x} + \beta\mathbf{y} \in V$$



Linear Algebra is everywhere

- Most world's phenomena involve complicated equations
- Computers can only do basic arithmetic
- ➔ Usually can't do the original equations, approximate by series of **linear equations**
- ➔ Model things as **linear spaces**

$$E(q) = \sum_{T} A_T \|\nabla q^T - \mathbf{w}^T\|^2 \rightarrow \min$$

$$E(V') = \sum_{i=1}^{nT} \|\delta_i - \mathcal{L}(\mathbf{v}'_i))\|^2 + \sum_{i=m}^{n} \|\mathbf{v}'_i - \mathbf{u}_i\|^2,$$

$$q^i_m = \sum_k \frac{1}{M_i - 1} \sum_{j \in O_m} X^{ji} \sum_{n=1}^{3} w^{ij}_{mn} q^j_n$$

$$\arg\min_{w_j} \frac{1}{2} \int_\Omega |\Delta w_j|^2 dV$$

$$E(q) = \sum_T A_T \|\nabla q^T - \mathbf{w}^T\|^2 \rightarrow \min$$

$$\widetilde{I}^{\,i}(\cdot) : \bigcup_{k=1}^{d_i - 1} \triangle^i_k \longrightarrow \mathbb{R}.$$

$$\widetilde{I}^{\,i}(\mu) = \langle \mu, \mu \rangle_{\mathbb{R}^3} = \langle \mu_1 \widetilde{\mathbf{x}}^i_k + \mu_2 \widetilde{\mathbf{x}}^i_{k+1}, \ \mu_1 \widetilde{\mathbf{x}}^i_k + \mu_2 \widetilde{\mathbf{x}}^i_{k+1} \rangle_{\mathbb{R}^3} = $$
$$= \mu_1^2 \, \widetilde{g}^i_{k,k} + 2\, \mu_1 \, \mu_2 \, \widetilde{g}^i_{k,k+1} + \mu_2^2 \, \widetilde{g}^i_{k+1,k+1},$$

$$\hat{\mathbf{x}}^i_2 = \langle \widetilde{\mathbf{x}}^i_2, \frac{\widetilde{\mathbf{x}}^i_1}{\|\widetilde{\mathbf{x}}^i_1\|} \rangle \frac{\widetilde{\mathbf{x}}^i_1}{\|\widetilde{\mathbf{x}}^i_1\|} + \langle \widetilde{\mathbf{x}}^i_2, \mathbf{n} \rangle \, \mathbf{n} + \langle \hat{\mathbf{x}}^i_2, \mathbf{N}^i \rangle \, \mathbf{N}^i =$$

Examples from everyday life

Weather forecasting

- Solve PDEs (partial differential equations) that model the physics of the atmosphere
- Unknowns: temperature, humidity, wind… at every point in Earth's atmosphere at a certain time

Temperature

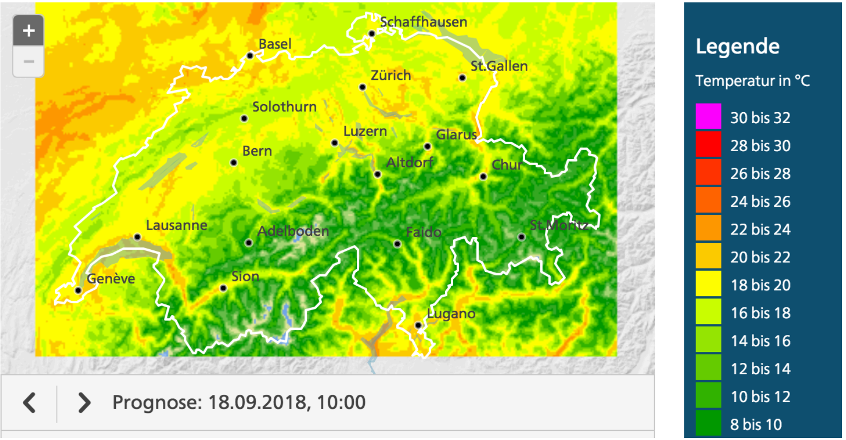


Image source: MeteoSwiss

Weather forecasting

- Analytical solution (formula) doesn't exist
- ➔ Discretization on a grid, numerical approximation
- **Huge** systems of linear equations

$$A\mathbf{x} = \mathbf{b}$$

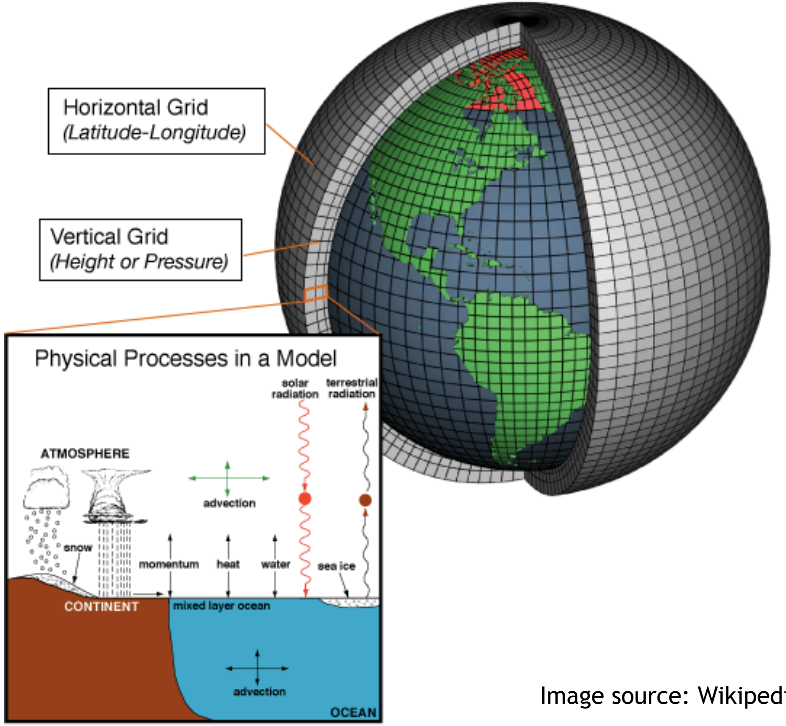


Image source: Wikipedia

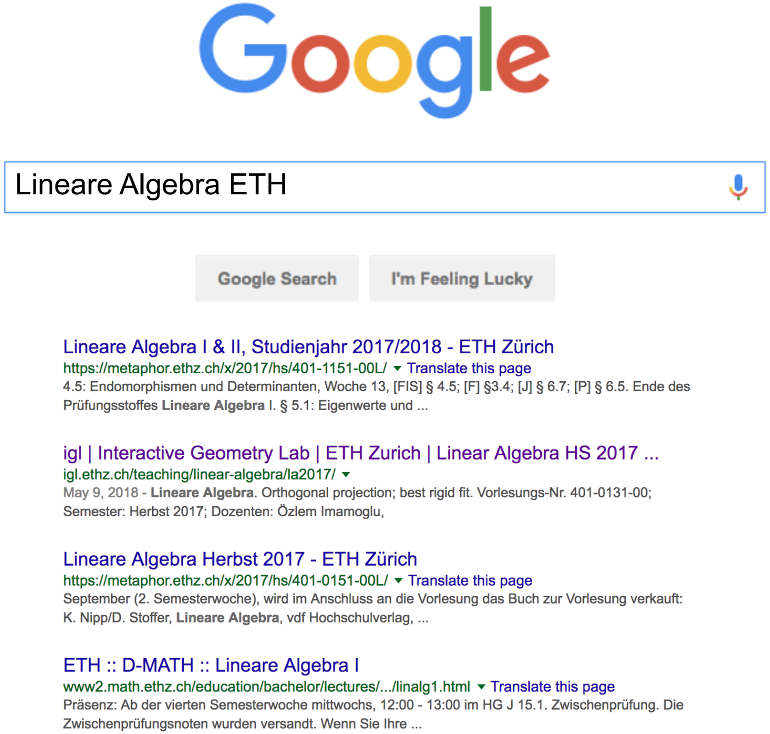
Weather forecasting

- Linear algebra done by supercomputers!
- CS challenge: how to solve huge linear equations, and fast

Some of the MeteoSwiss supercomputers at CSCS, Lugano

Google search engine

- Web crawler “reads” the Internet pages and indexes by keywords
- User enters keyword, search engine retrieves pages containing it
- In what order to present the found pages??



Google search engine – PageRank

- **PageRank** algorithm sorts search results by **importance**
- Importance of a page = how many other important pages link to it

$$\text{PageRank}(u) = \sum_{v:\ v \text{ links to } u} \frac{\text{PageRank(v)}}{\#\text{ links from v}}$$

- PageRanks of all webpages? Eigenvalue problem! $A\mathbf{u} = \lambda\mathbf{u}$
- We will learn about it in the 2nd half of the semester

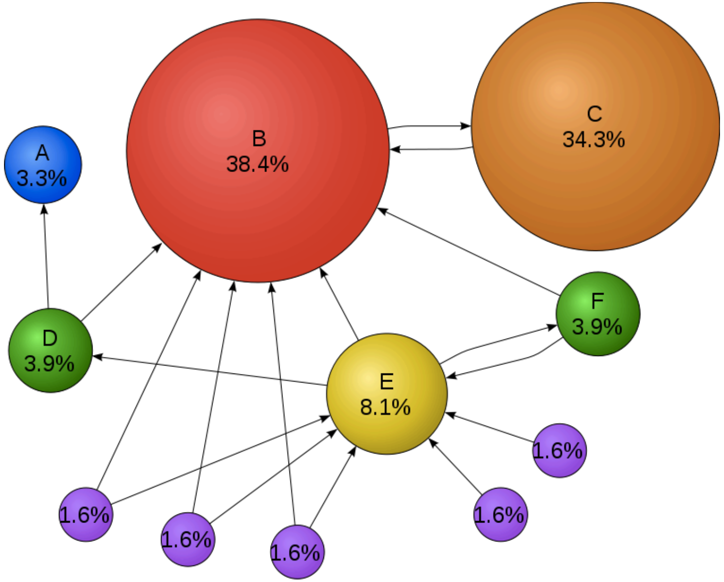
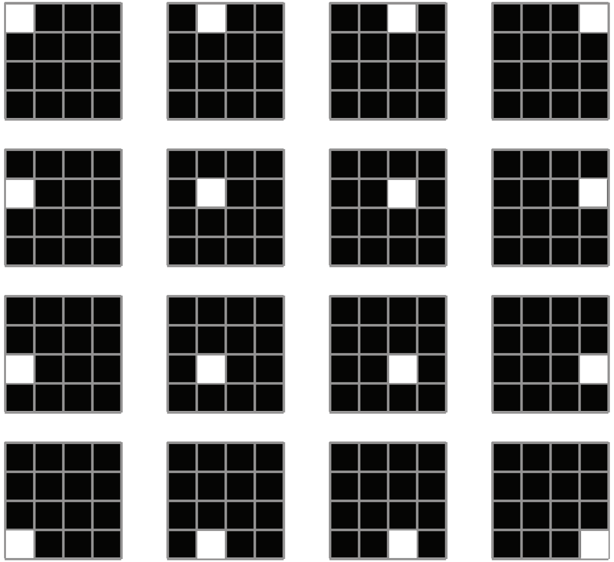


Image source: Wikipedia

Digital image representation

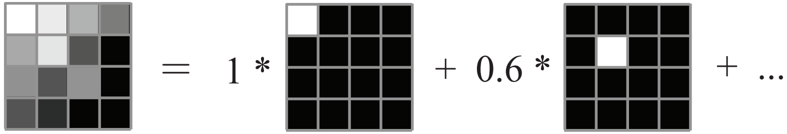
- Images are vectors!
- The image on the right:
  - 2272 x 1704 pixels
  - pixel = (R,G,B)-value
  - this image is a **11,614,464-dimensional vector**

Images as vectors



The standard basis for 4x4 grayscale images
16 vectors

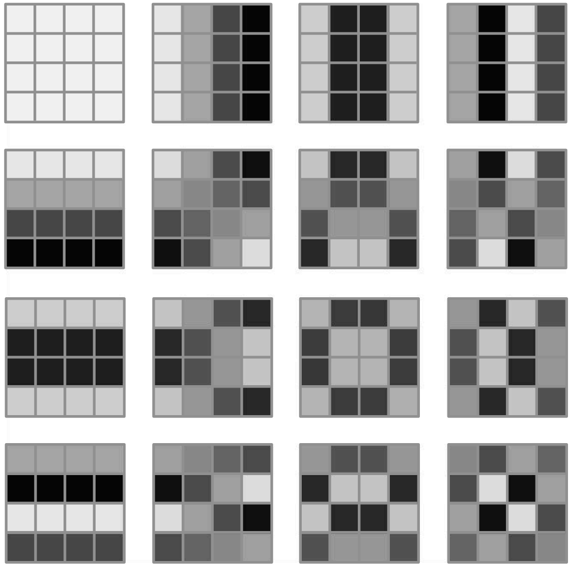
Any 4x4 grayscale image is a
linear combination of this standard basis



$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \ldots + \alpha_n \mathbf{b}_n$$

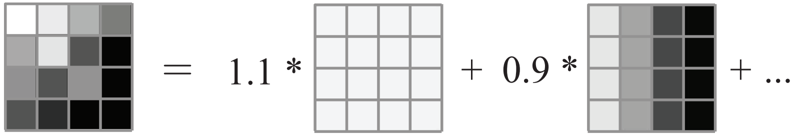
Need to store all $\alpha_1, \alpha_2, \ldots, \alpha_n$

JPEG image compression



The 4x4 DCT (discrete cosine) basis
16 vectors

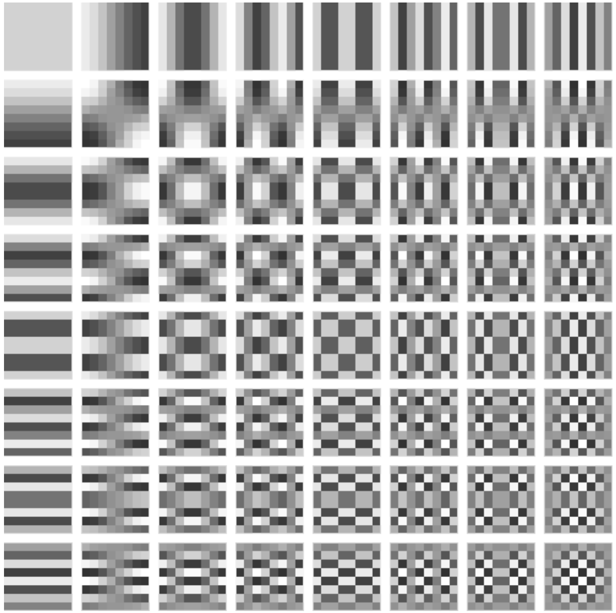
Any 4x4 grayscale image is **also** a linear combination of that basis!



$$\mathbf{x} = \beta_1 \mathbf{c}_1 + \beta_2 \mathbf{c}_2 + \ldots + \beta_n \mathbf{c}_n$$

For “natural” images we can omit all but a few first β

JPEG image compression



The 8x8 DCT (discrete cosine) basis
64 vectors

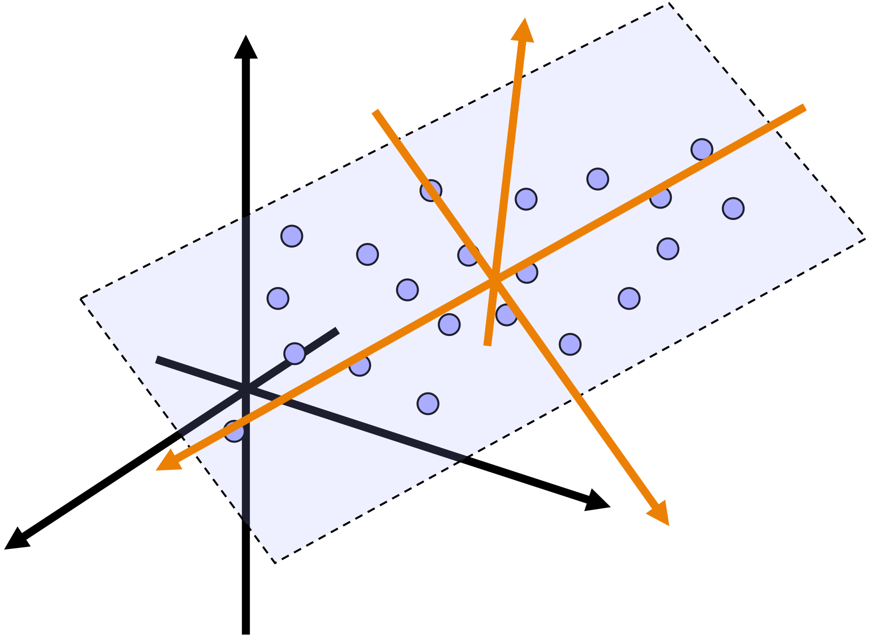
Any 8x8 grayscale image is a
linear combination of that basis!

$$\mathbf{x} = \beta_1 \mathbf{c}_1 + \beta_2 \mathbf{c}_2 + \ldots + \beta_n \mathbf{c}_n$$

For “natural” images we can omit all but a few first β

JPEG image compression

- Images are vectors in a (high-dimensional) space
- Different coordinate systems = different bases
- JPEG image compression: project onto a lower-dimensional linear space



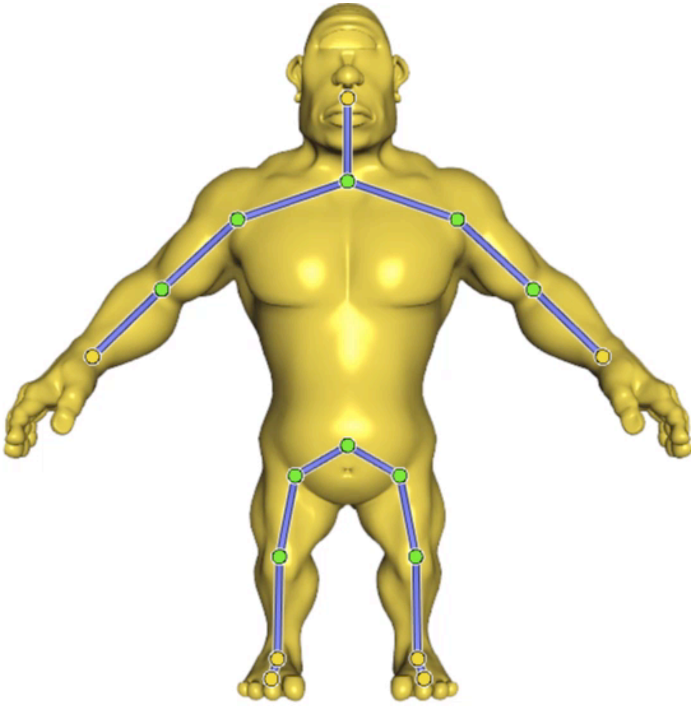
Computer animation

- How do virtual characters move?

Excerpt from “Big Buck Bunny”, open Blender movie

Computer animation

- Artist designs key poses for skeleton
- Collection of linear transformations in 3D space
- Automatic interpolation over the character’s surface and over time



Linear Algebra is fundamental

Enjoy the class!

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