G22.3033-008, Spring 2010

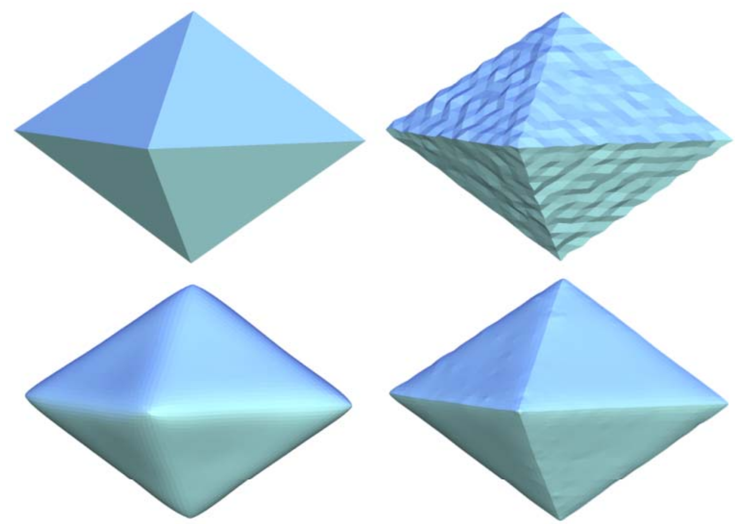
Differential Geometry Primer

Discrete Theory in a nutshell
Curves

Acknowledgement: many thanks to Eitan Grinspun for the DDG slides and course notes

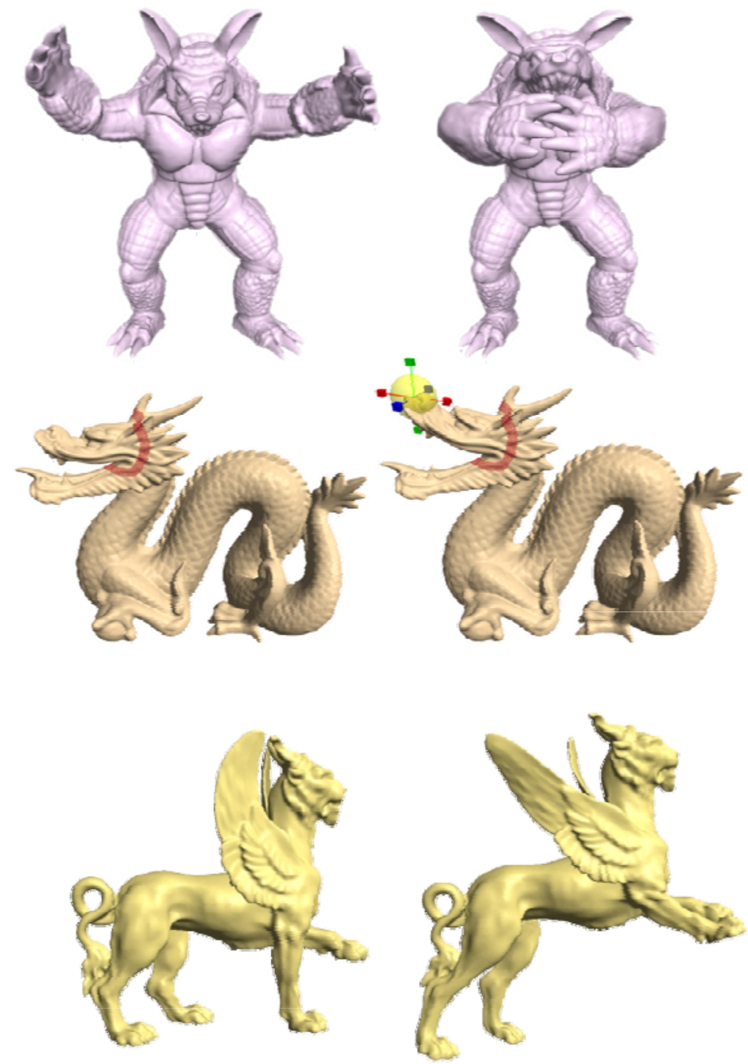
Motivation

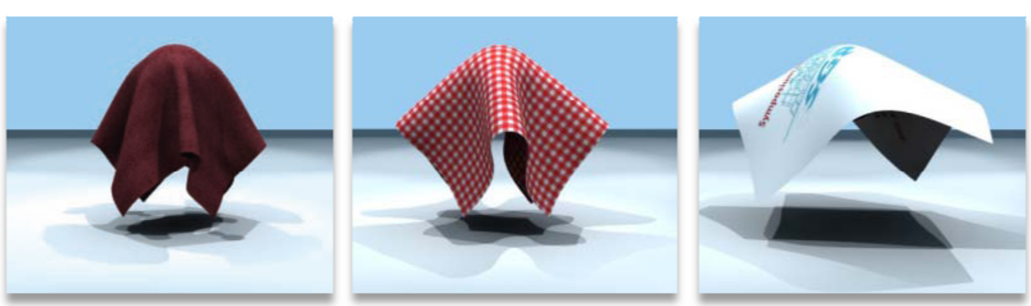
- Geometry processing: understand geometric characteristics, e.g.
  - smoothness



Motivation

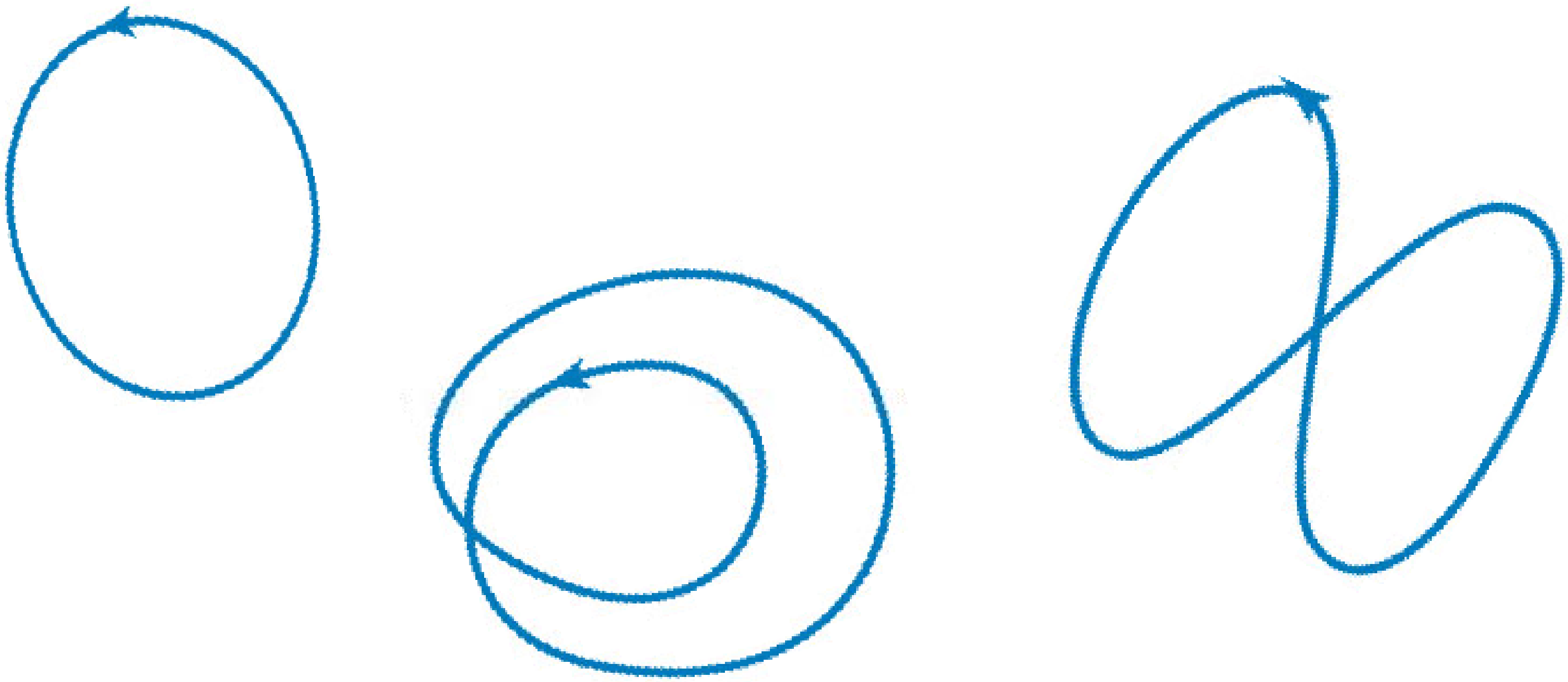
- Geometry processing: understand geometric characteristics, e.g.
  - smoothness
  - how shapes deform





Curves

smooth definition



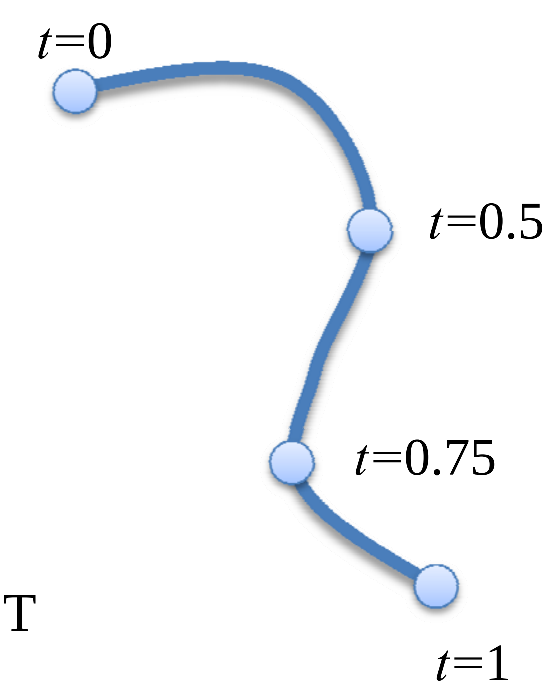
Curves

smooth definition

- Curves are 1-dimensional parameterizations

$$\mathbf{p}: \mathrm{R} \to \mathrm{R}^d, \quad d = 1, 2, 3, \ldots$$

$$t \to \mathbf{p}(t)$$



  - Planar curve: $\mathbf{p}(t) = (x(t), y(t))^{\mathrm{T}}$
  - Space curve: $\mathbf{p}(t) = (x(t), y(t), z(t))^{\mathrm{T}}$

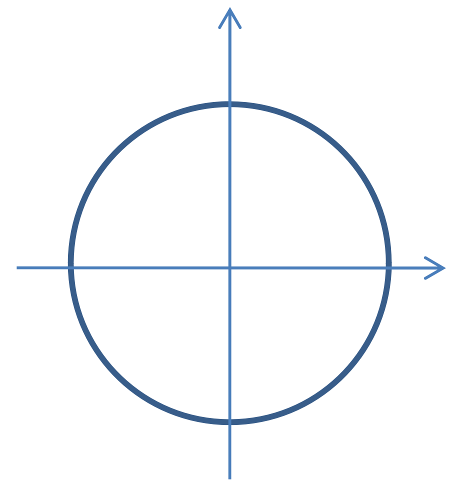
Parametric Curves

Examples

- Circle in 2D

$$\mathbf{p}(t) = (r\cdot\cos(t),\ r\cdot\sin(t))^{\mathrm{T}}$$

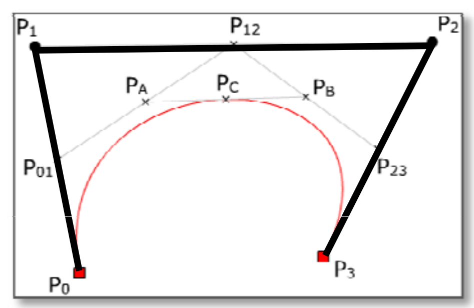
$$t \in [0, 2\pi)$$



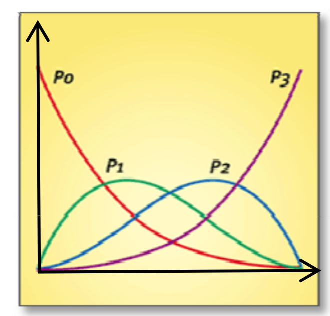
- Bézier curve

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{p}_i\, B_i^n(t)$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



Curve and control polygon



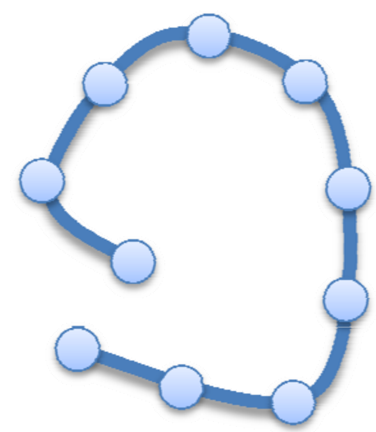
Basis functions

Curves

Arc length parameterization

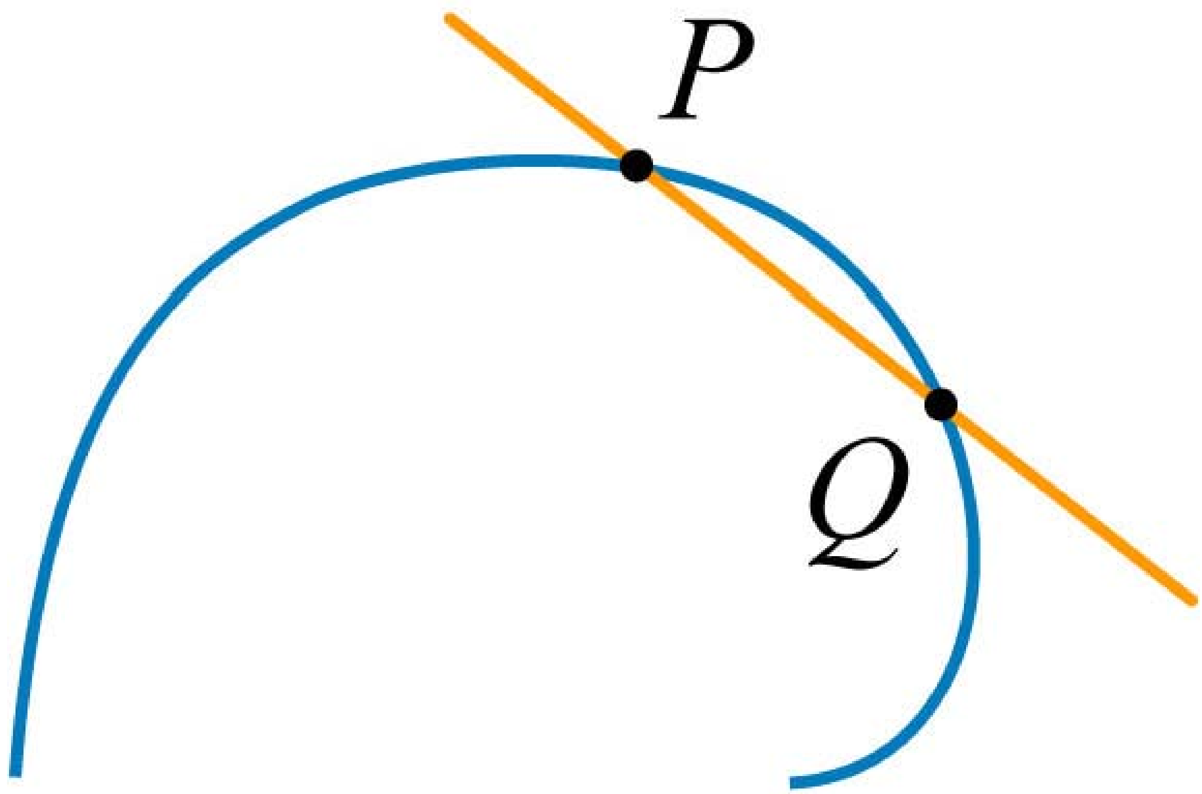
- Equal pace of the parameter along the curve

- $len\,(\mathbf{p}(t_1), \mathbf{p}(t_2)) = |t_1 - t_2|$



Secant

- A line through two points on the curve.



Secant

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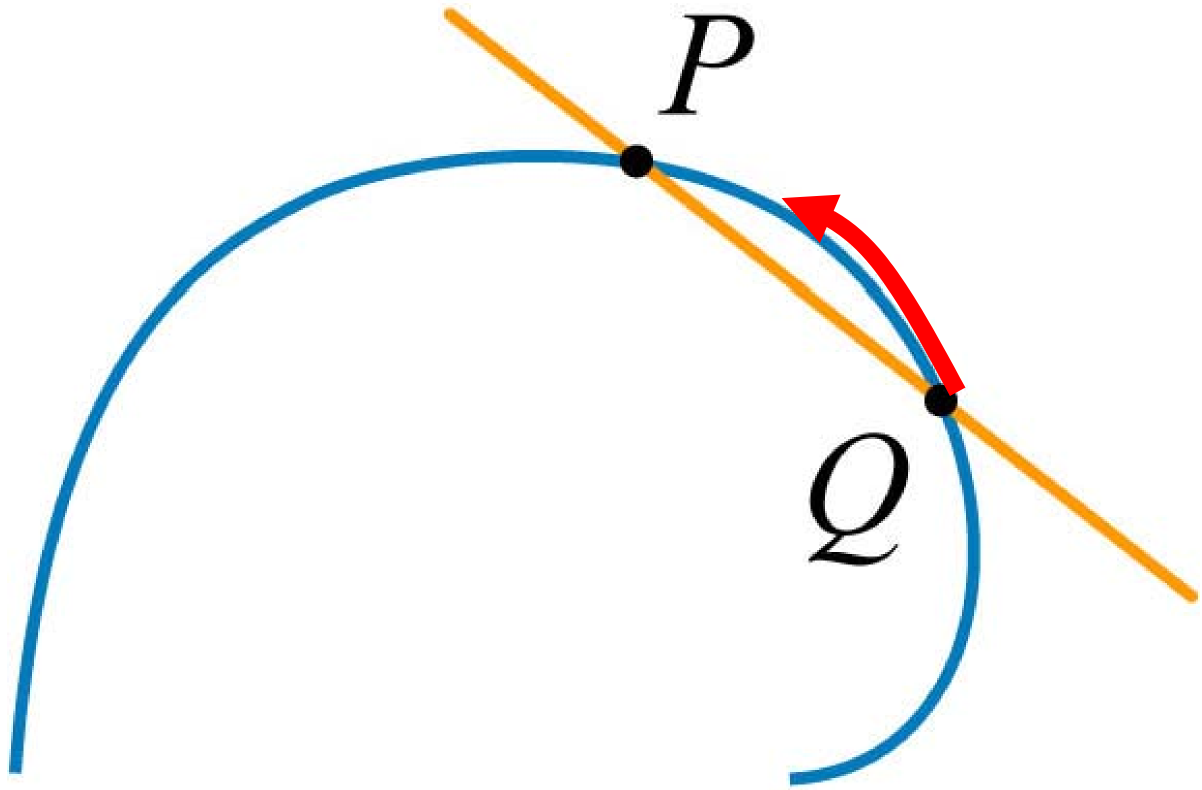
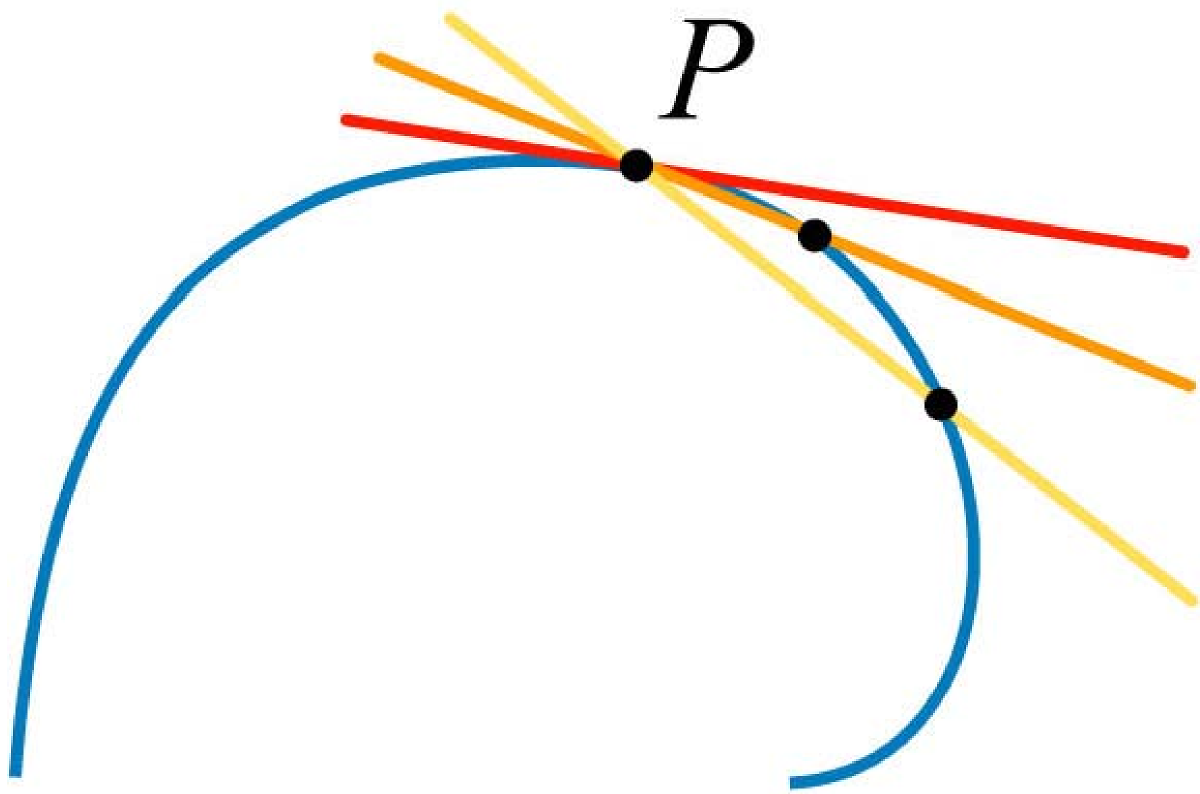


Tangent

- The limiting secant as the two points come together.

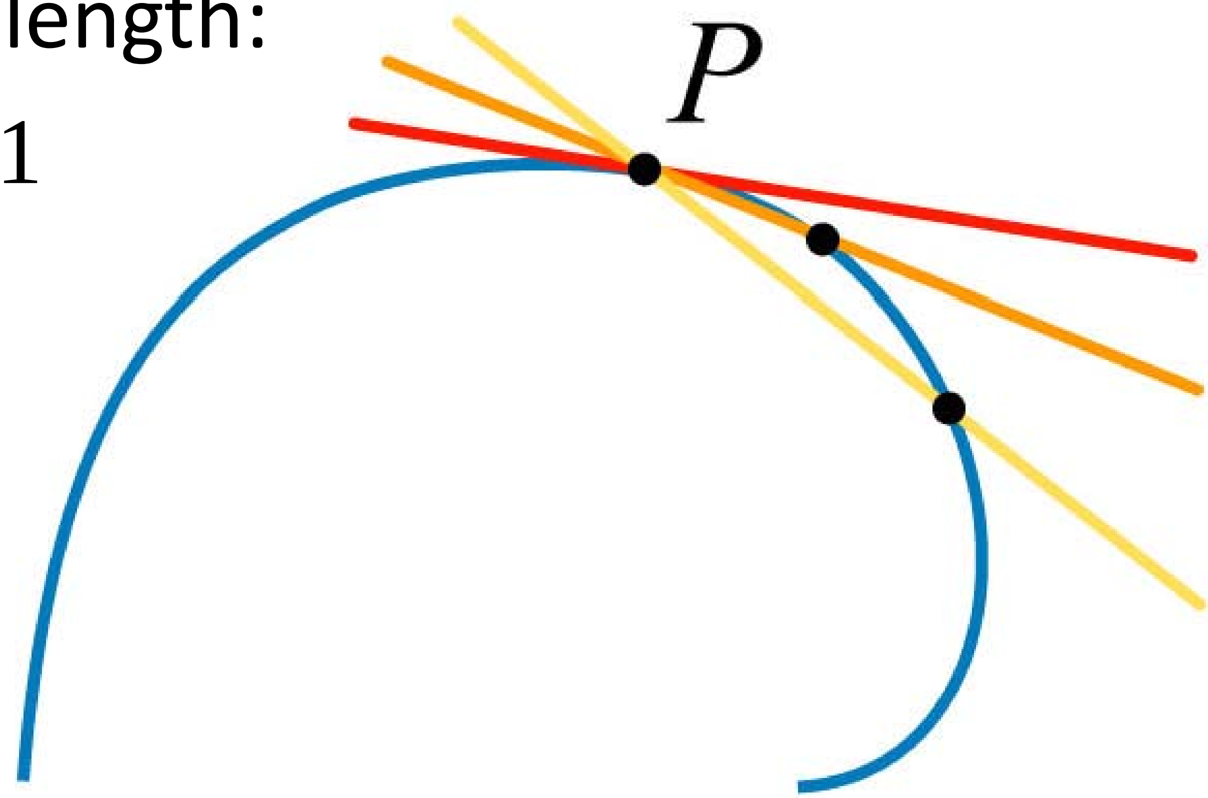


Secant and tangent

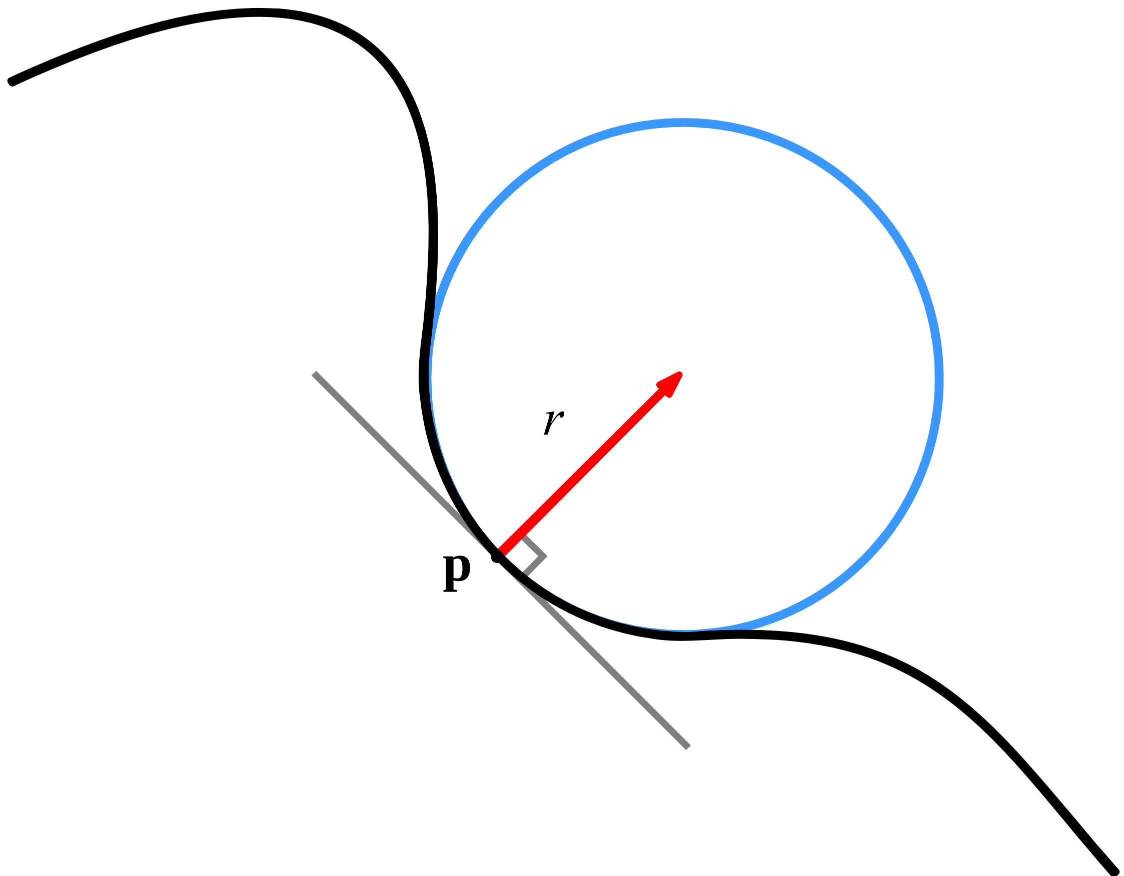
parametric form

- Secant: $\mathbf{p}(t) - \mathbf{p}(s)$
- Tangent: $\mathbf{p}'(t) = (x'(t), y'(t), \ldots)^{\mathrm{T}}$
- If t is arc-length:

  $\|\mathbf{p}'(t)\| = 1$

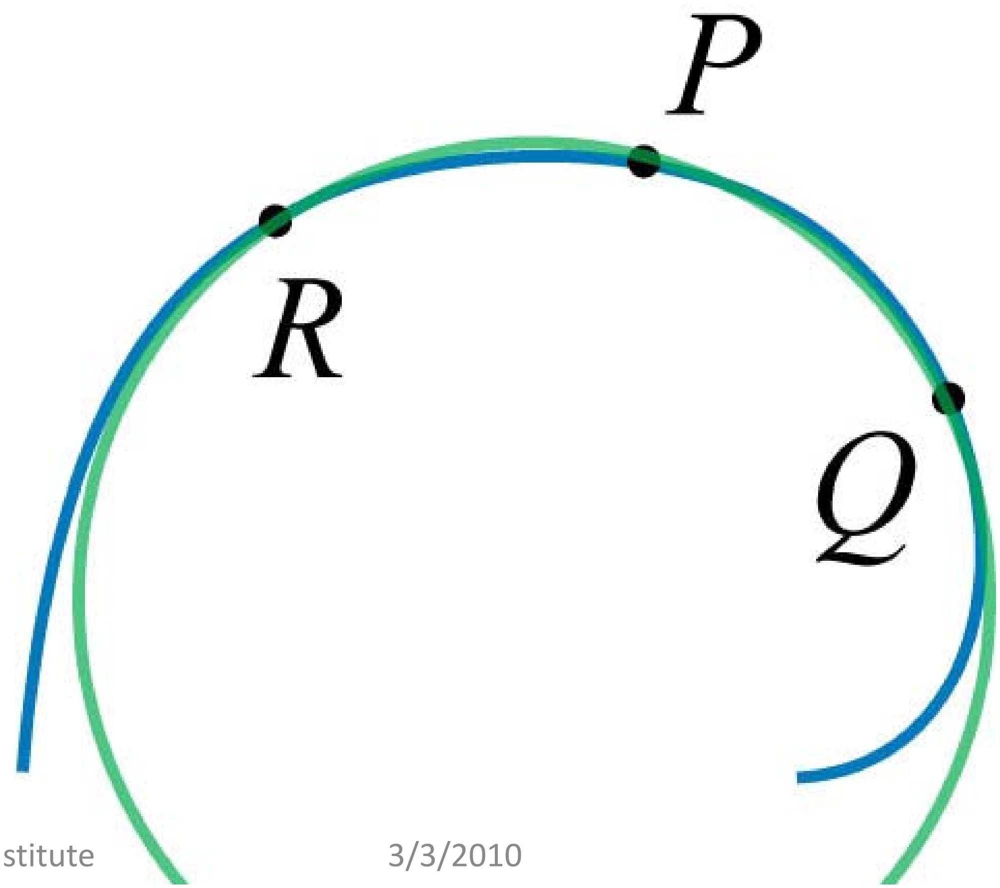


Tangent, normal, radius of curvature



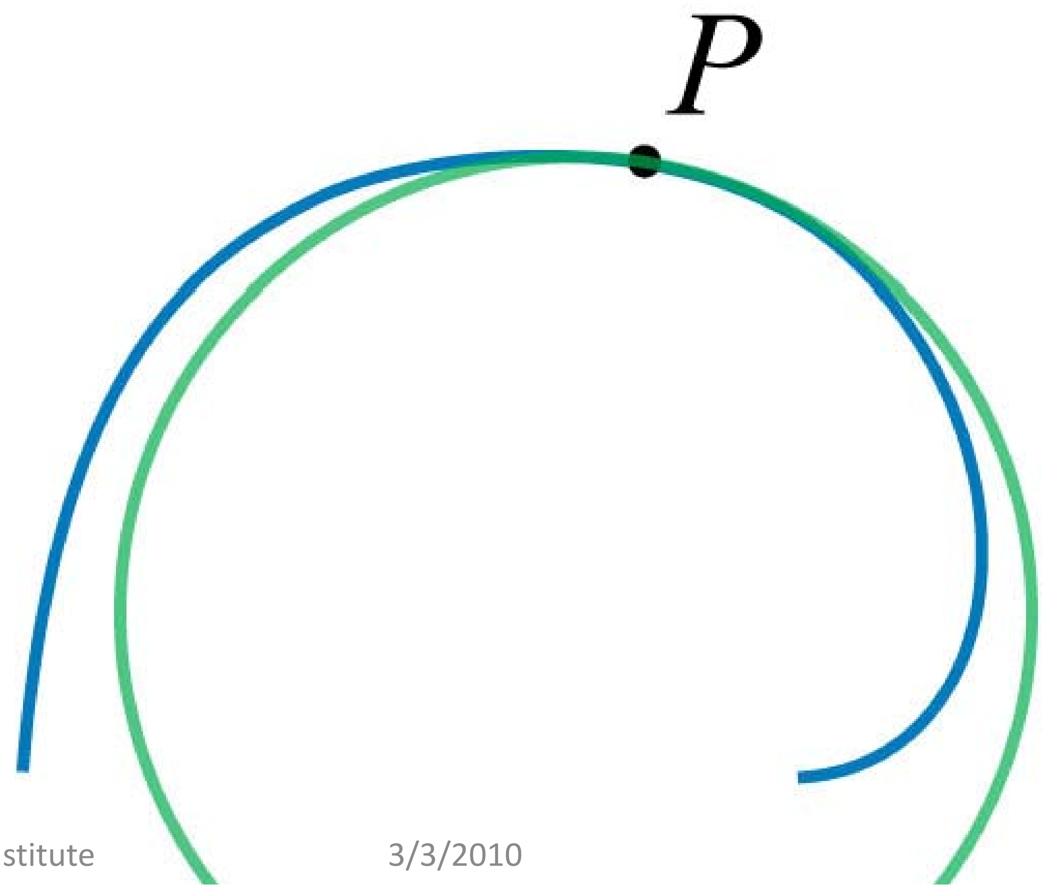
Circle of curvature

- Consider the circle passing through three points on the curve…

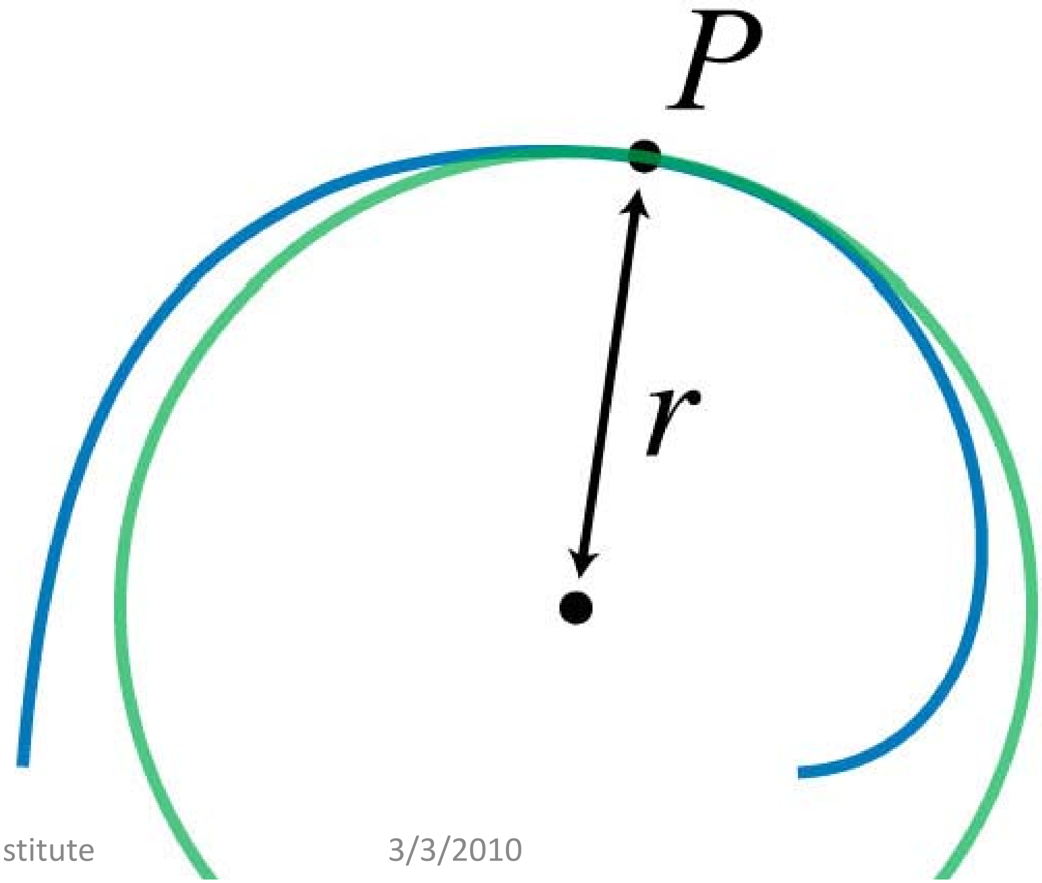


Circle of curvature

- …the limiting circle as three points come together.



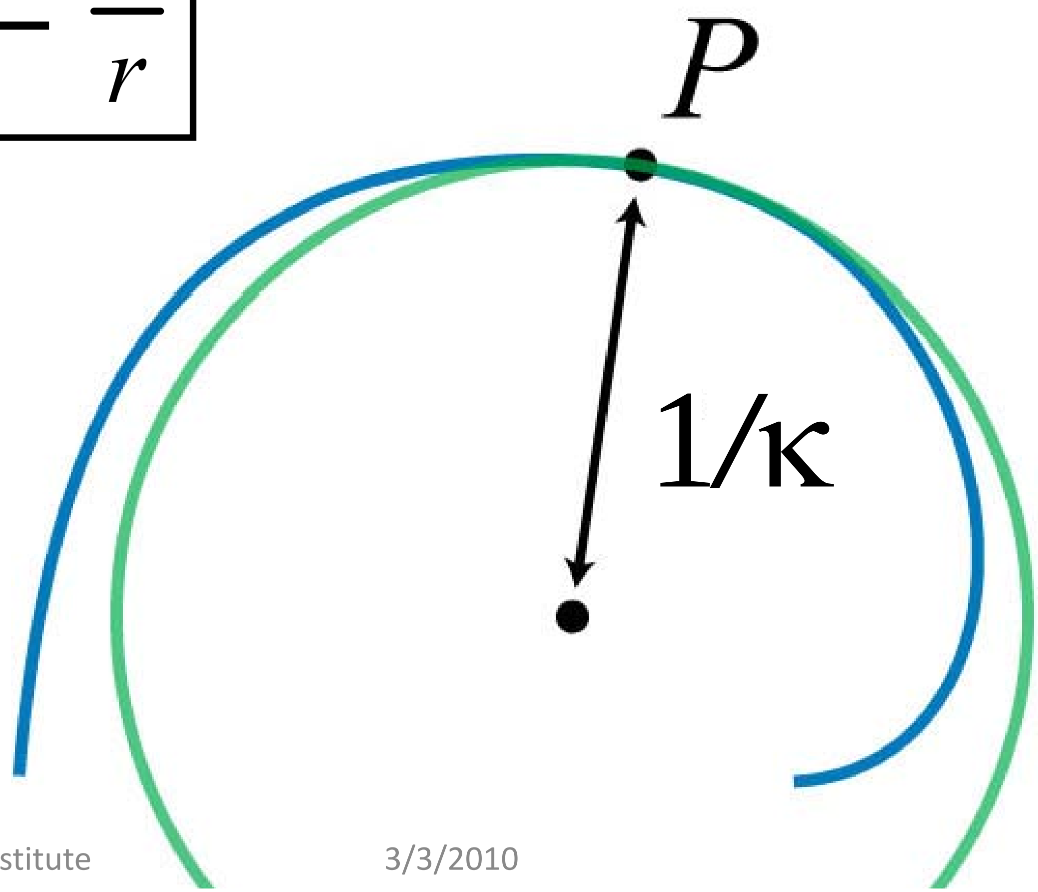
Radius of curvature, r



Radius of curvature, $r = 1/\kappa$

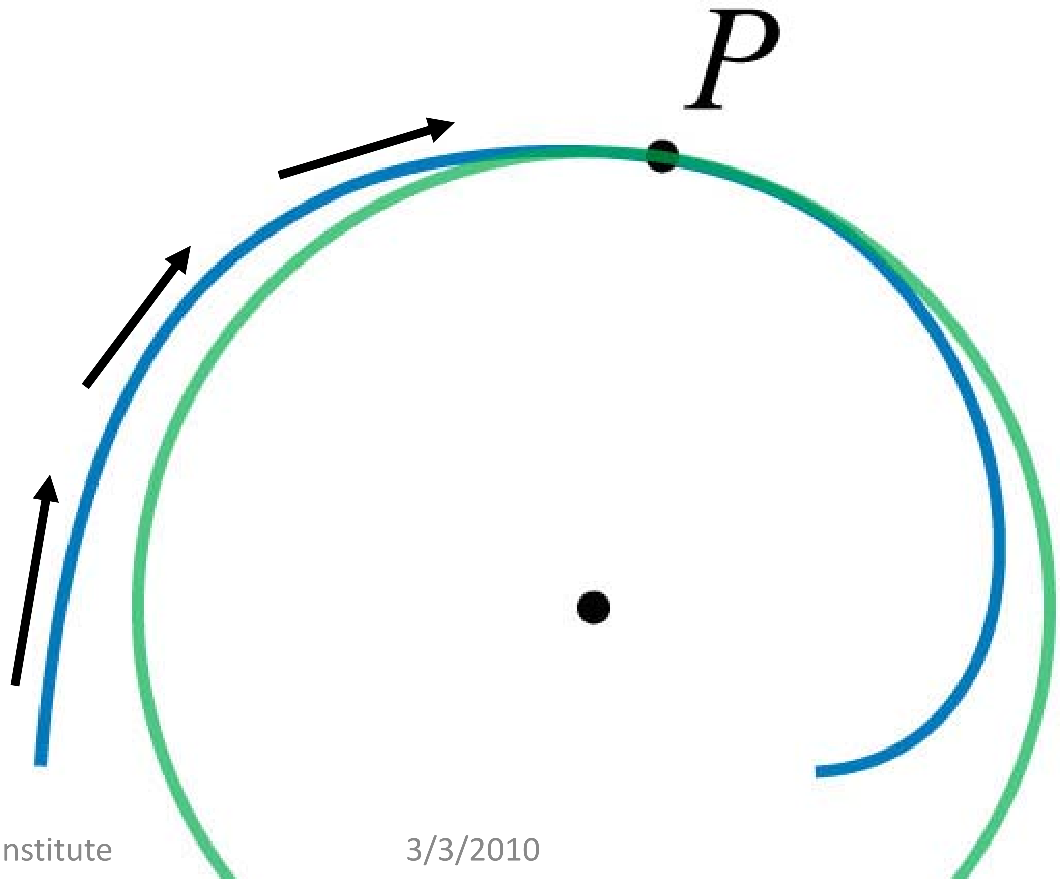
Curvature

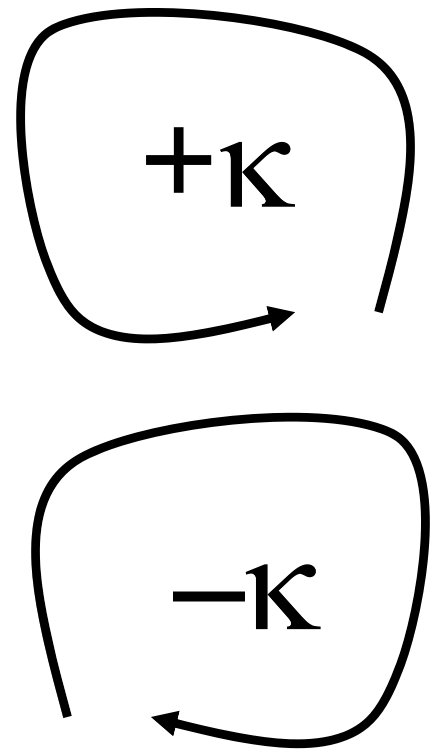
$$\boxed{\kappa = \frac{1}{r}}$$



Signed curvature

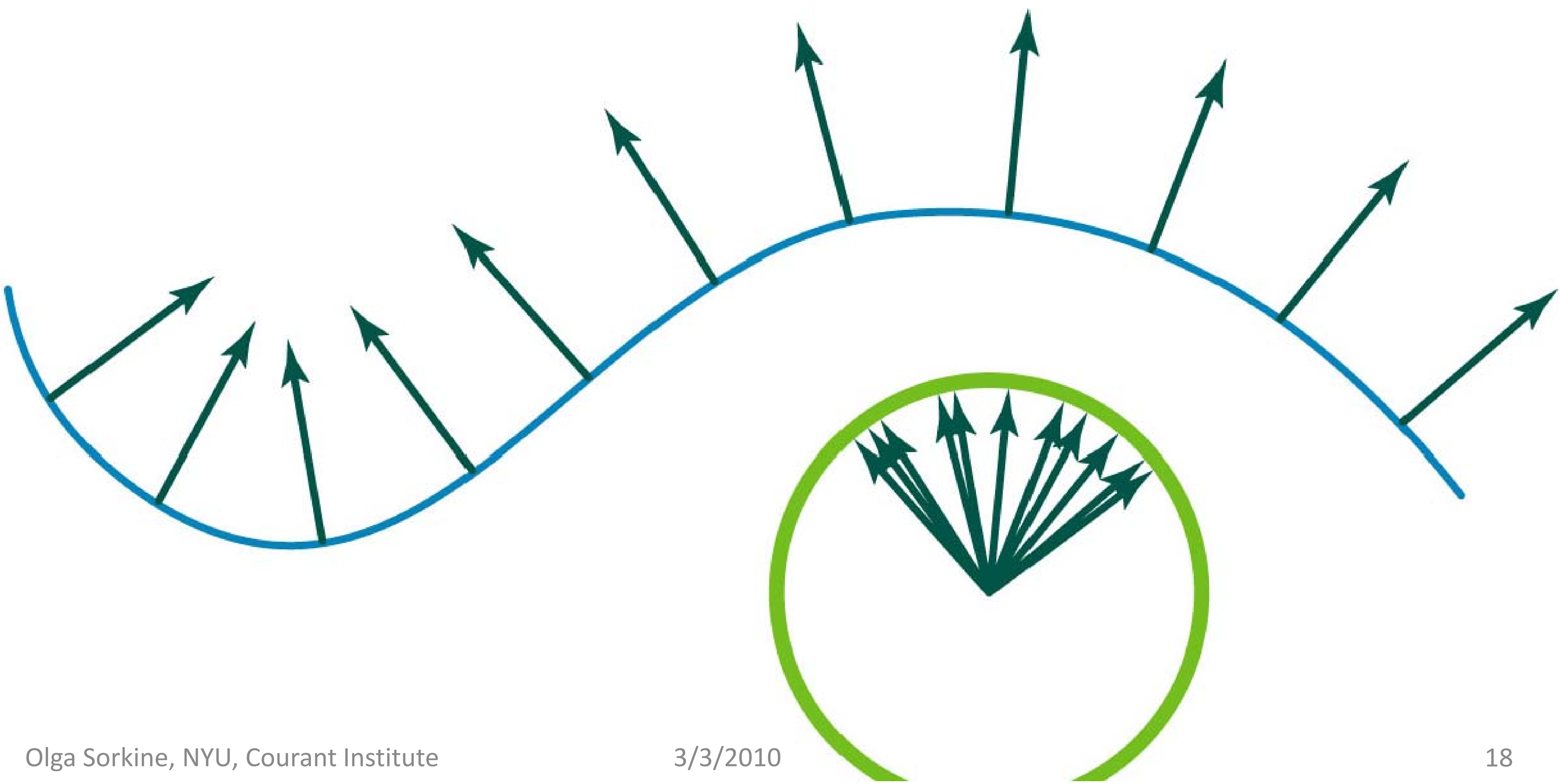
- Sense of traversal along curve.





Gauss map, $\hat{\mathbf{n}}(\mathbf{p})$

- Point on curve maps to point on unit circle.

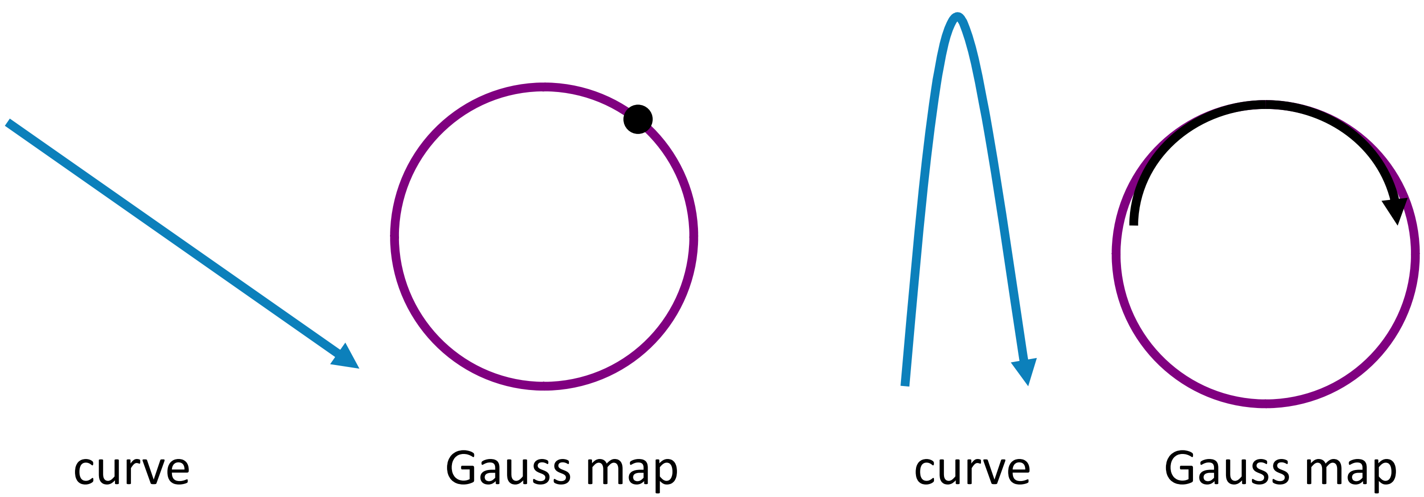


Curvature = change in normal direction

- Absolute curvature (assuming arc length t)

$$\kappa = \left\| \hat{\mathbf{n}}'(t) \right\|$$

- Parameter-free view: via the Gauss map

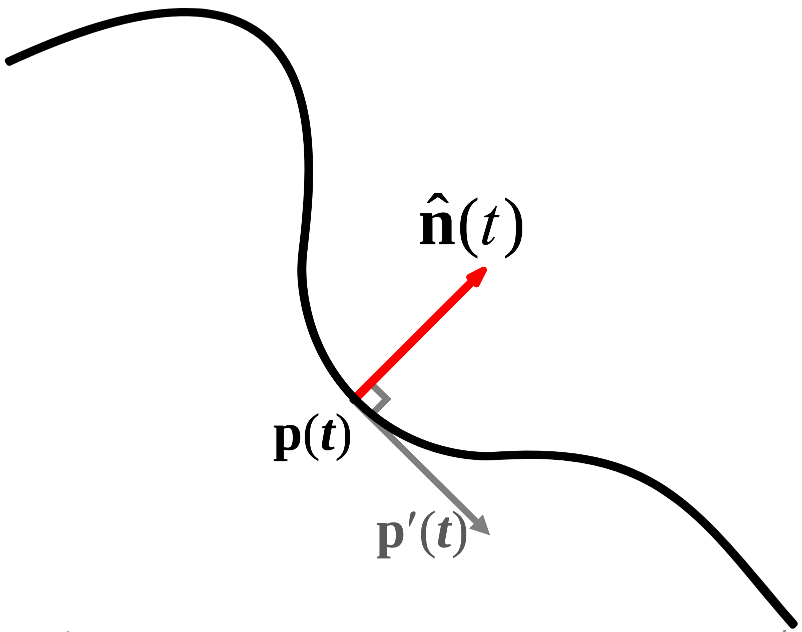


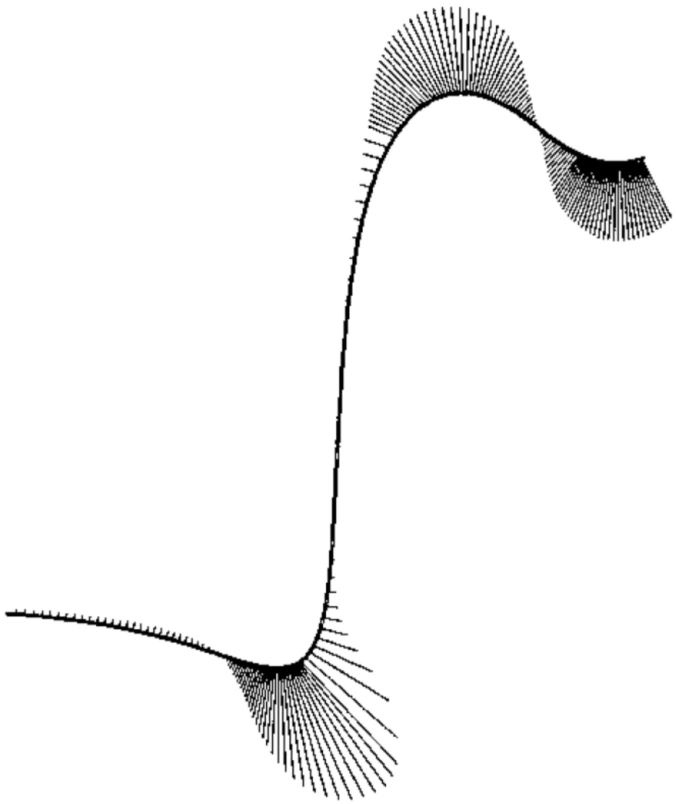
Curvature normal

parametric form

- Assume t is arc-length parameter

$$\mathbf{p}''(t) = \kappa \hat{\mathbf{n}}(t)$$





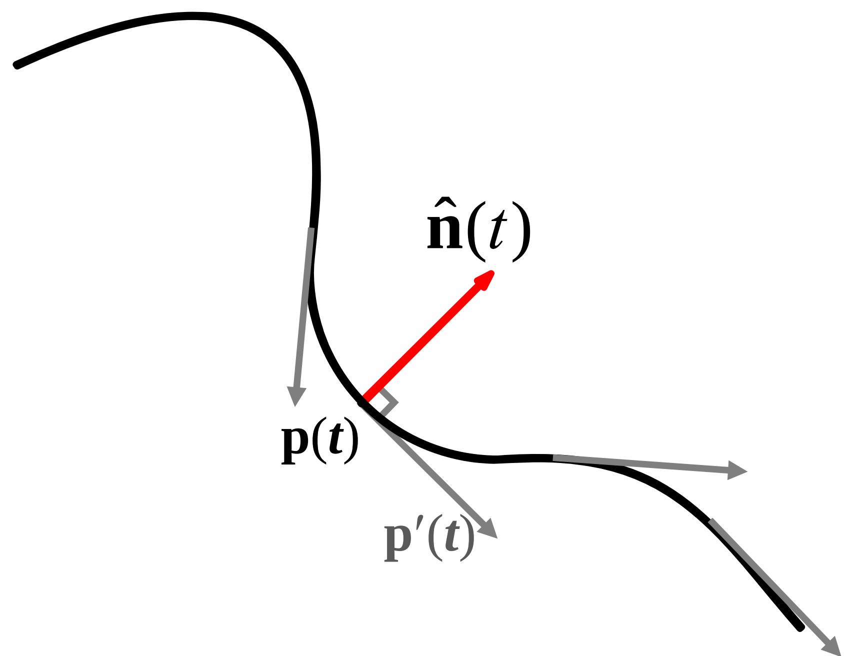
[Kobbelt and Schröder]

Curvature normal

parametric form

- Note: if the parameter has constant speed, it only changes along the normal direction
- In other words,

$$\mathbf{p}''(t) \perp \mathbf{p}'(t)$$



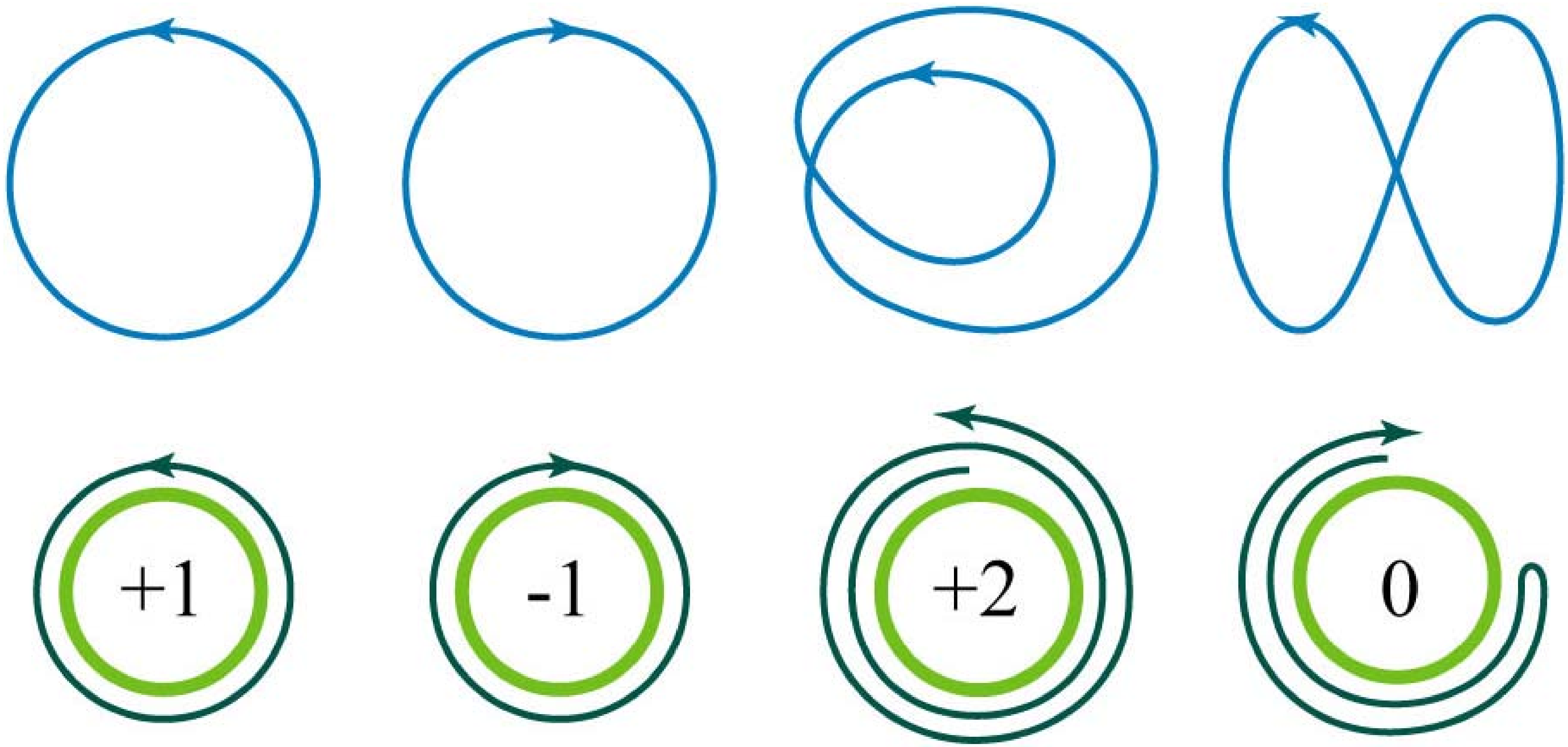
$$\langle \mathbf{p}'(t), \mathbf{p}'(t) \rangle = 1 \quad / \, \textit{differentiate both sides}$$

$$\langle \mathbf{p}''(t), \mathbf{p}'(t) \rangle + \langle \mathbf{p}'(t), \mathbf{p}''(t) \rangle = 0$$

$$\langle \mathbf{p}''(t), \mathbf{p}'(t) \rangle = 0$$

Turning number, k

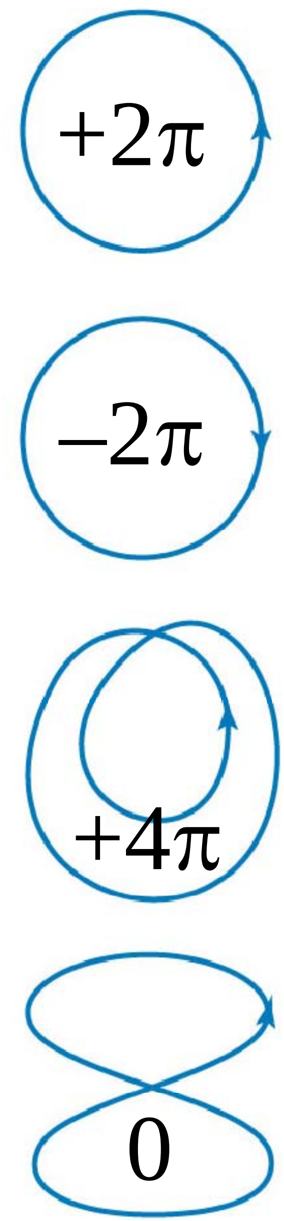
- Number of orbits in Gaussian image.



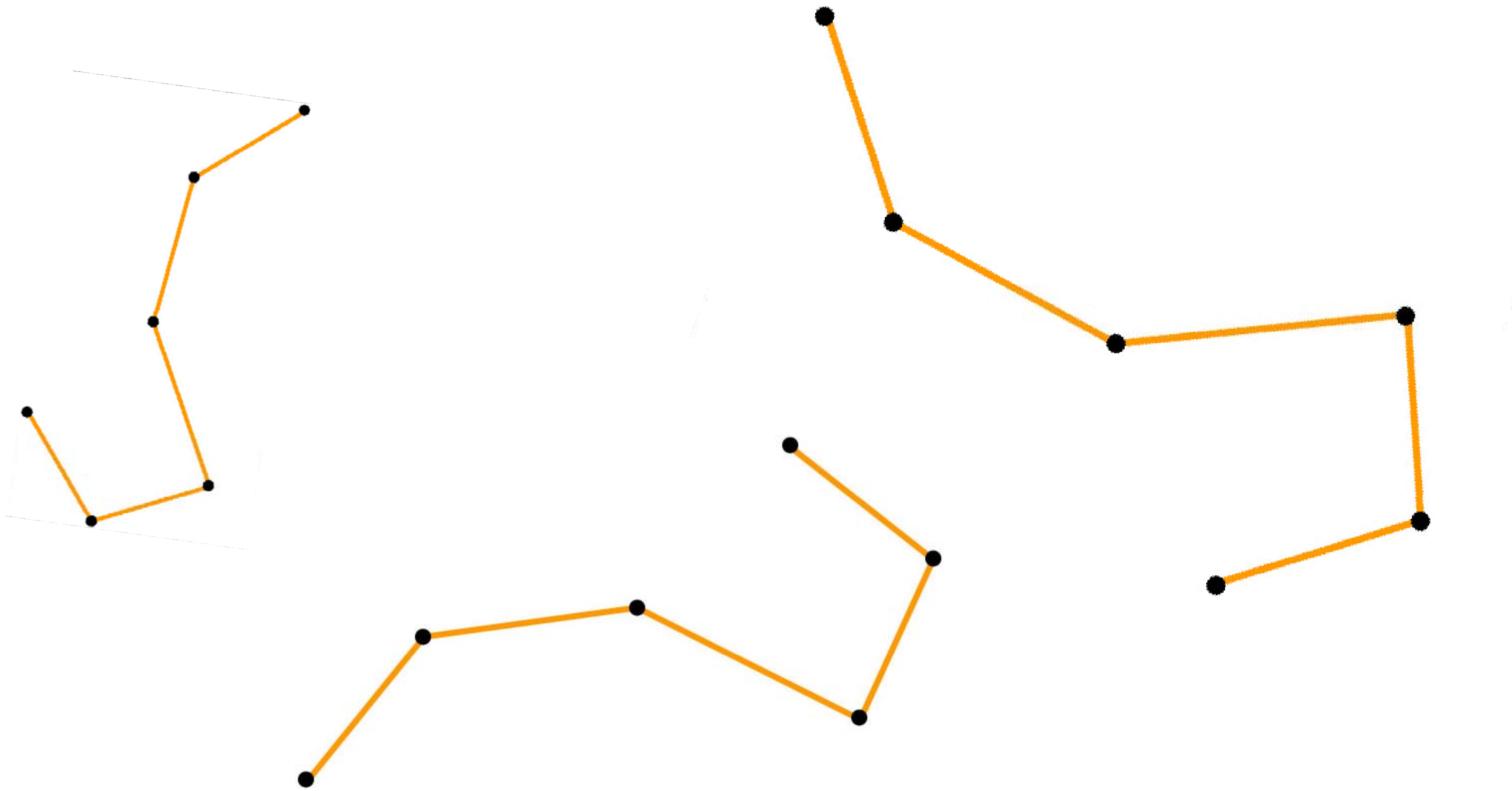
Turning number theorem

$$\boxed{\int_{\Omega} \kappa \, ds = 2\pi k}$$

- For a closed curve,
  the integral of curvature is
  an integer multiple of 2π.



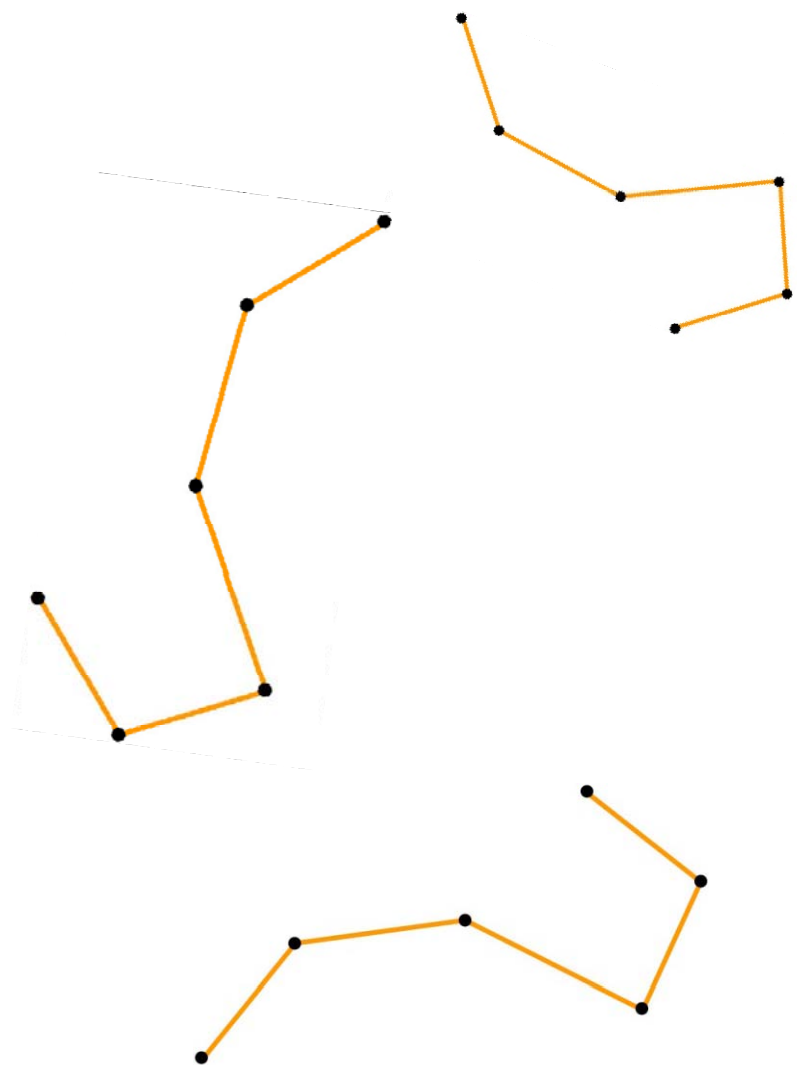
Discrete planar curves



Discrete planar curves

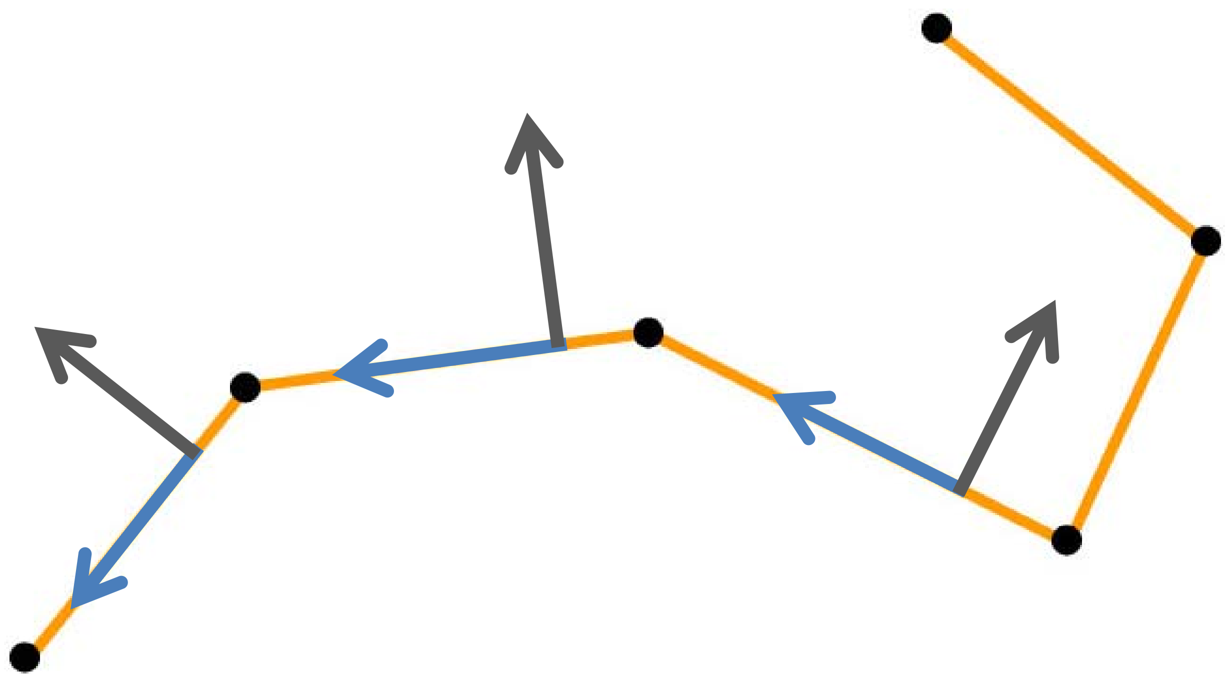
- Piecewise linear curves
- Not smooth at vertices
- Can’t take derivatives

- Generalize notions from the smooth world for the discrete case!



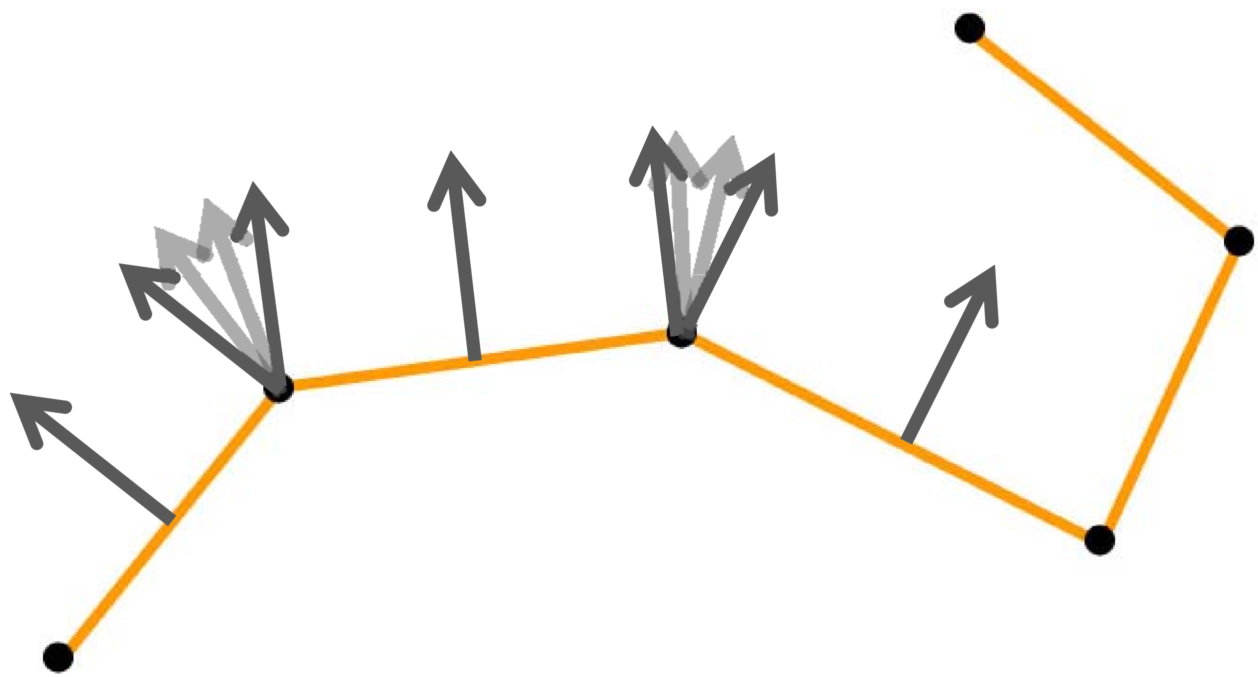
Tangents, normals

- For any point on the edge, the tangent is simply the unit vector along the edge and the normal is the perpendicular vector



Tangents, normals

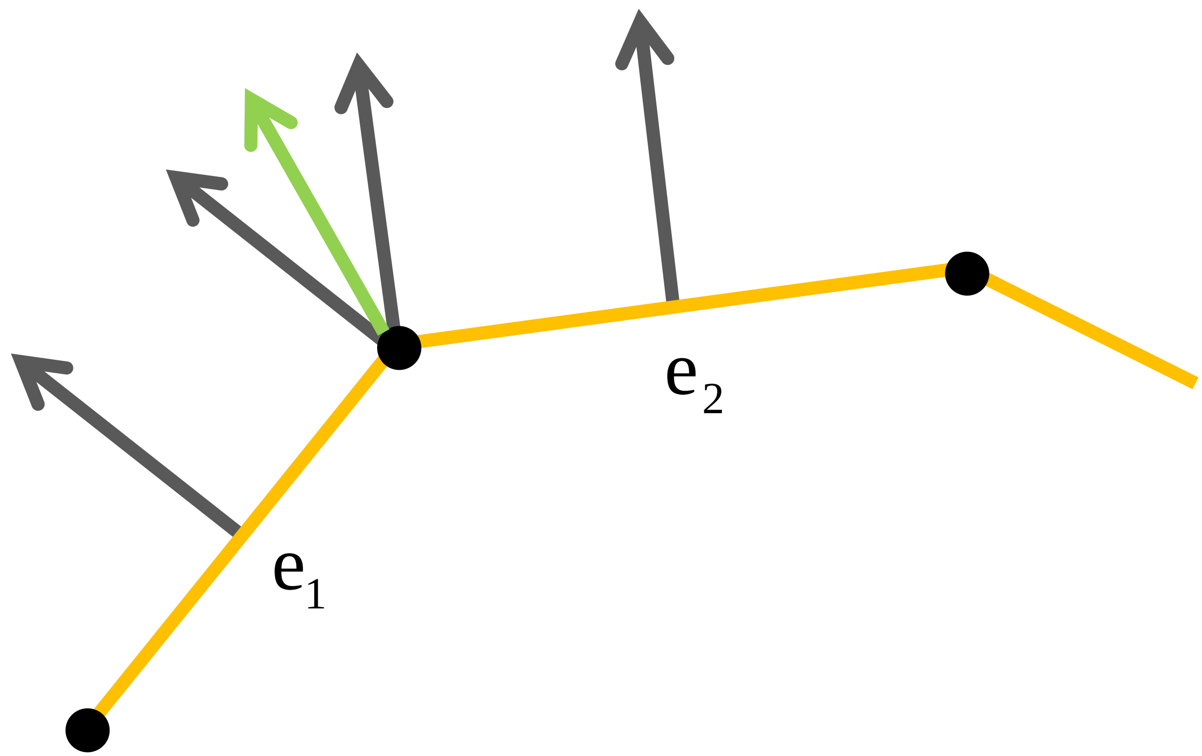
- For vertices, we have many options



Tangents, normals

- Can choose to average the adjacent edge normals

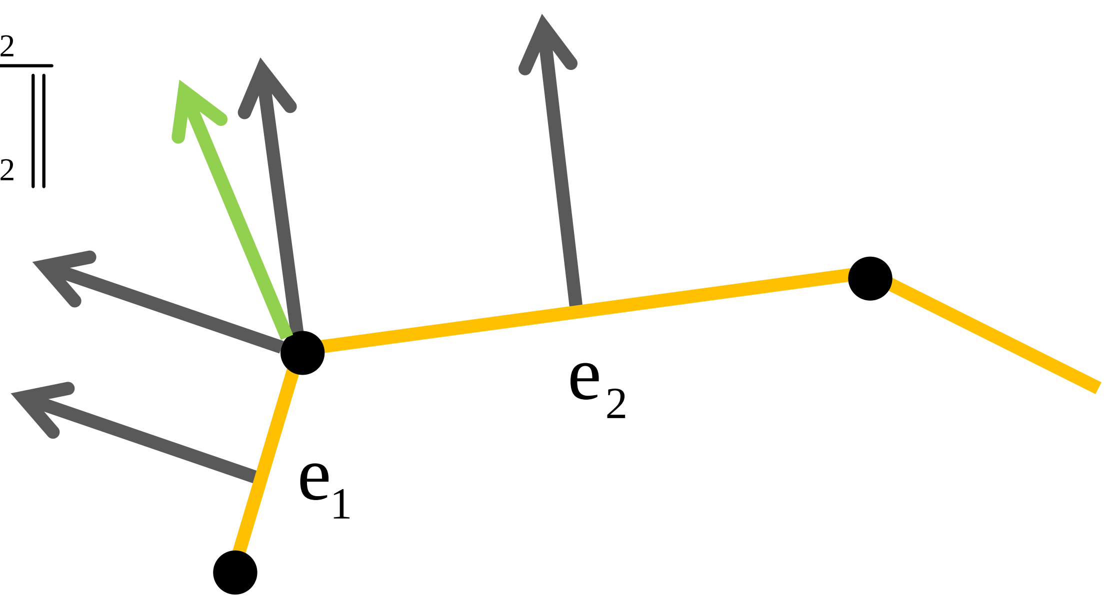
$$\hat{\mathbf{n}}_{\mathrm{v}} = \frac{\hat{\mathbf{n}}_{\mathrm{e}_1} + \hat{\mathbf{n}}_{\mathrm{e}_2}}{\left\| \hat{\mathbf{n}}_{\mathrm{e}_1} + \hat{\mathbf{n}}_{\mathrm{e}_2} \right\|}$$



Tangents, normals

- Weight by edge lengths

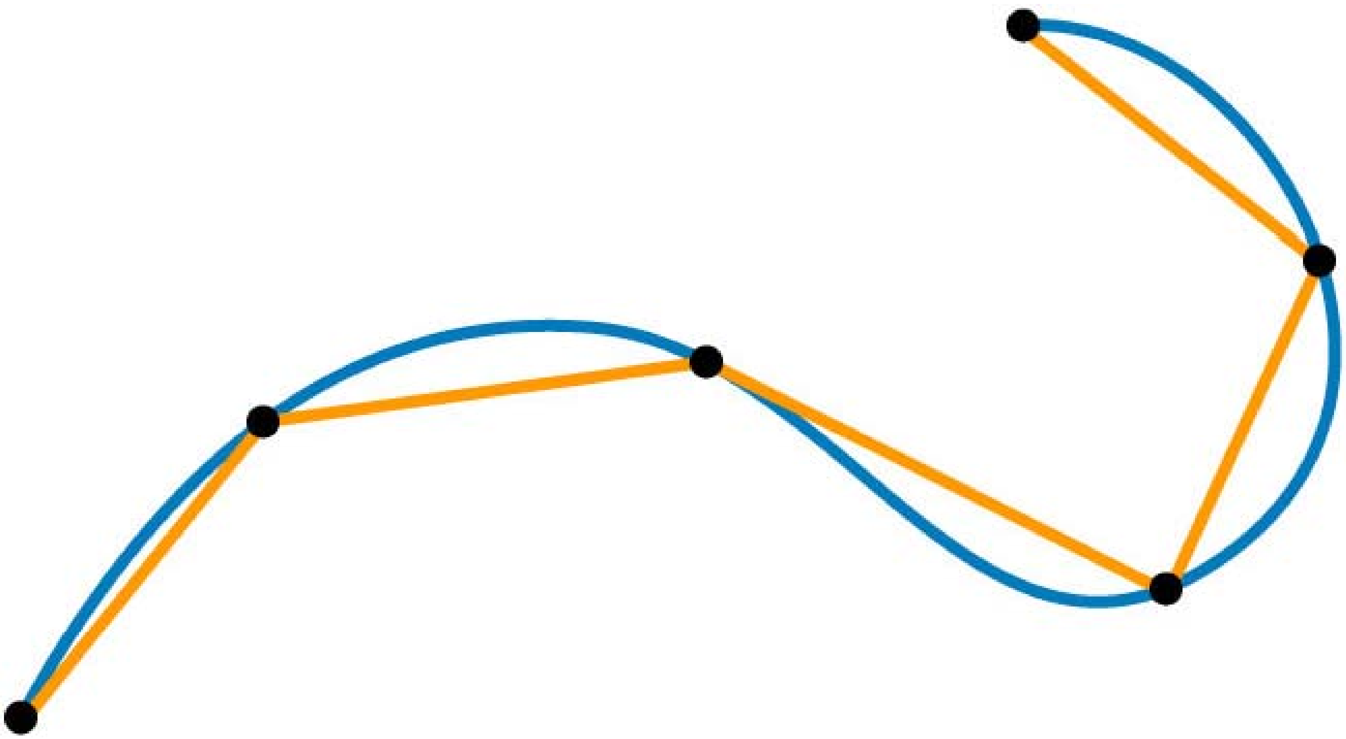
$$\hat{\mathbf{n}}_{\mathrm{v}} = \frac{|\mathrm{e}_1| \cdot \hat{\mathbf{n}}_{\mathrm{e}_1} + |\mathrm{e}_2| \cdot \hat{\mathbf{n}}_{\mathrm{e}_2}}{\left\| |\mathrm{e}_1| \cdot \hat{\mathbf{n}}_{\mathrm{e}_1} + |\mathrm{e}_2| \cdot \hat{\mathbf{n}}_{\mathrm{e}_2} \right\|}$$



Inscribed polygon, p

connection between discrete and smooth

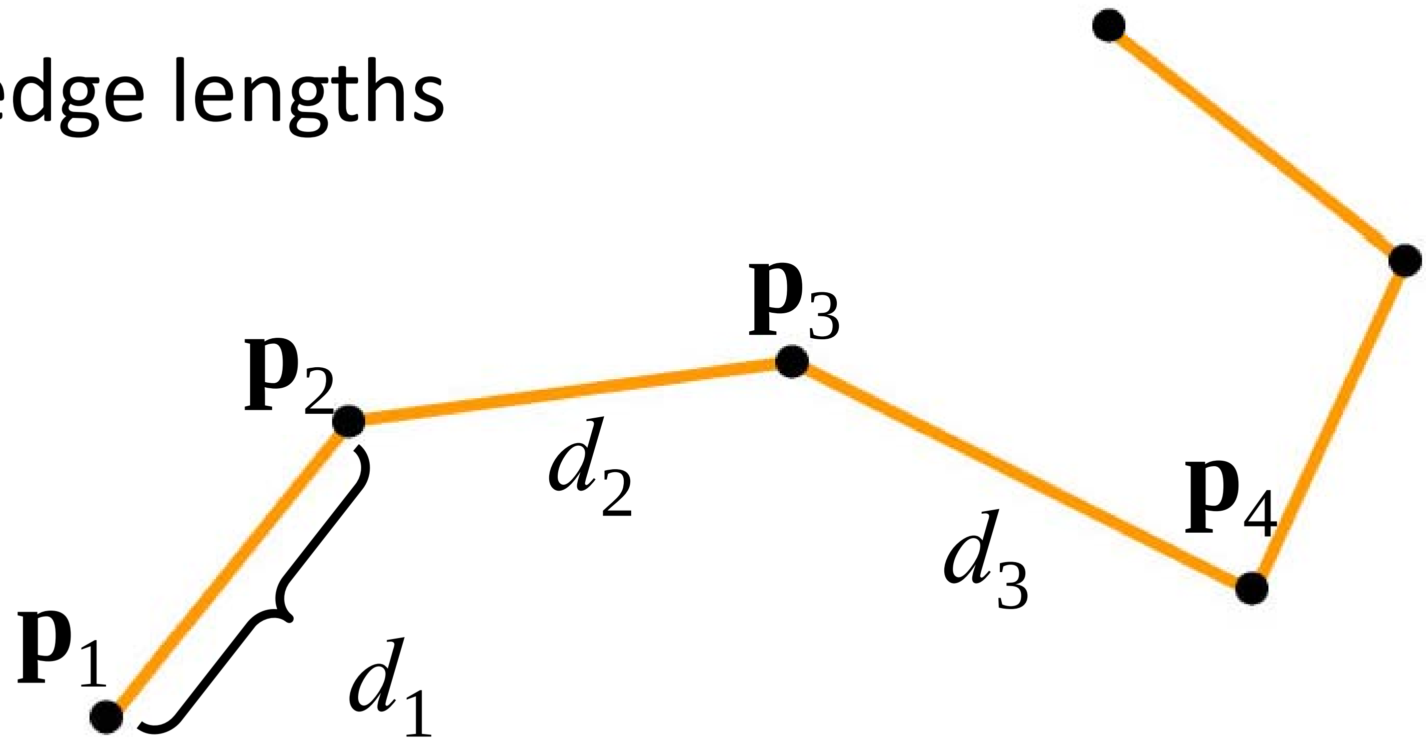
- Finite number of vertices each lying on the curve, connected by straight edges.



The length of a discrete curve

$$\boxed{\operatorname{len}(p)=\sum_{i=1}^{n} d_i=\sum_{i=1}^{n+1}\left\|\mathbf{p}_{i+1}-\mathbf{p}_i\right\|}$$

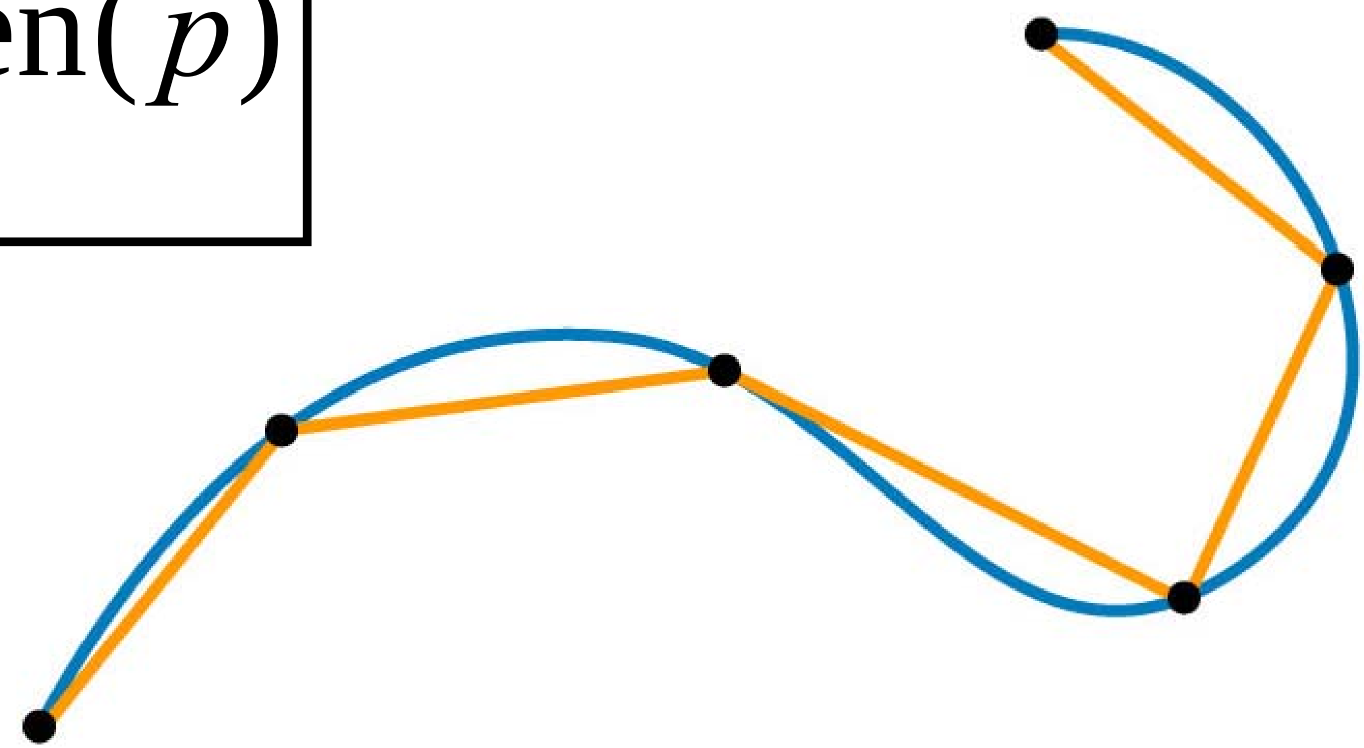
- Sum of edge lengths



The length of a continuous curve

- Length of longest of all inscribed polygons.

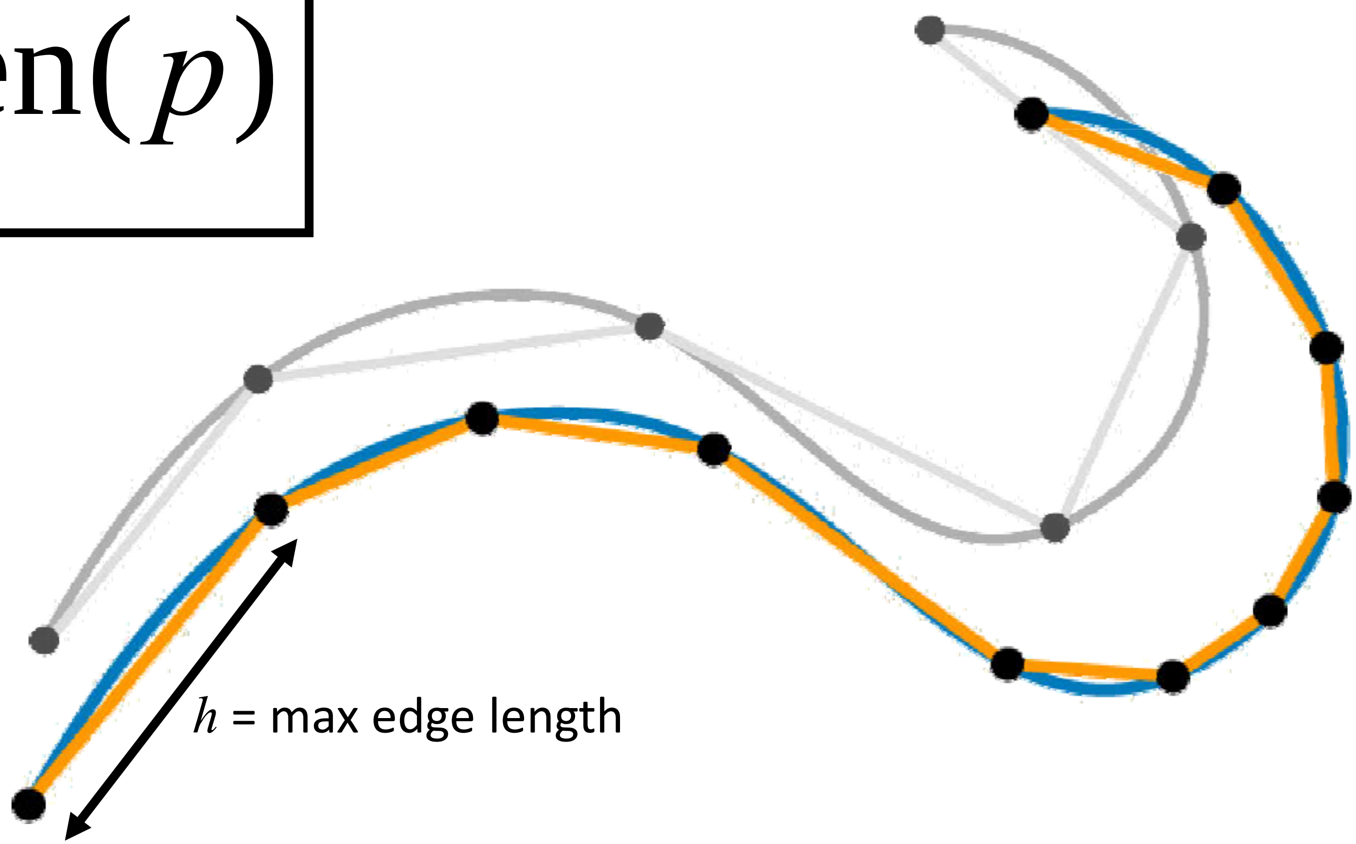
$$\boxed{\sup_{p} \operatorname{len}(p)}$$



The length of a continuous curve

- …or take limit over a refinement sequence

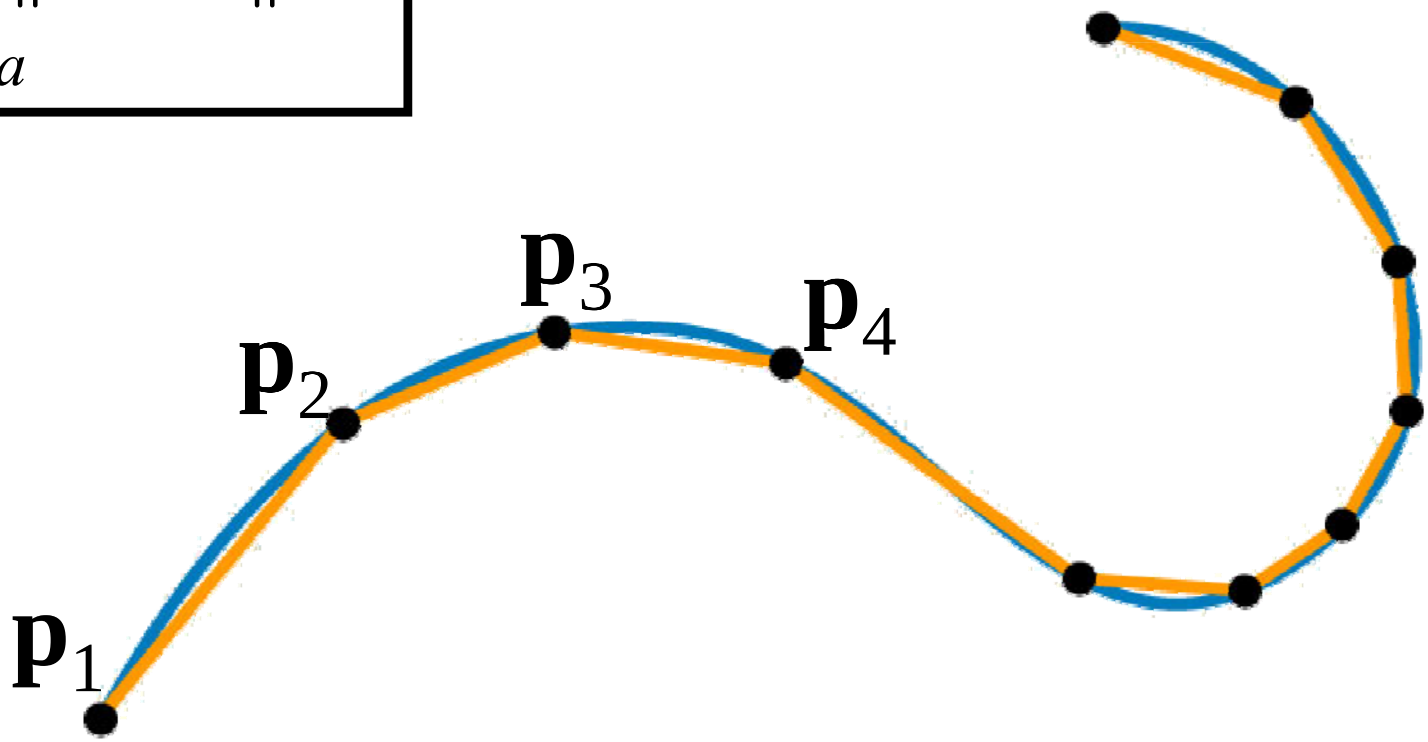
$$\boxed{\lim_{h\to 0} \operatorname{len}(p)}$$



The length of a continuous curve

- In the continuous form:

$$\boxed{\text{len} = \int_{s=a}^{b} \left\| \mathbf{p}'(s) \right\| ds}$$



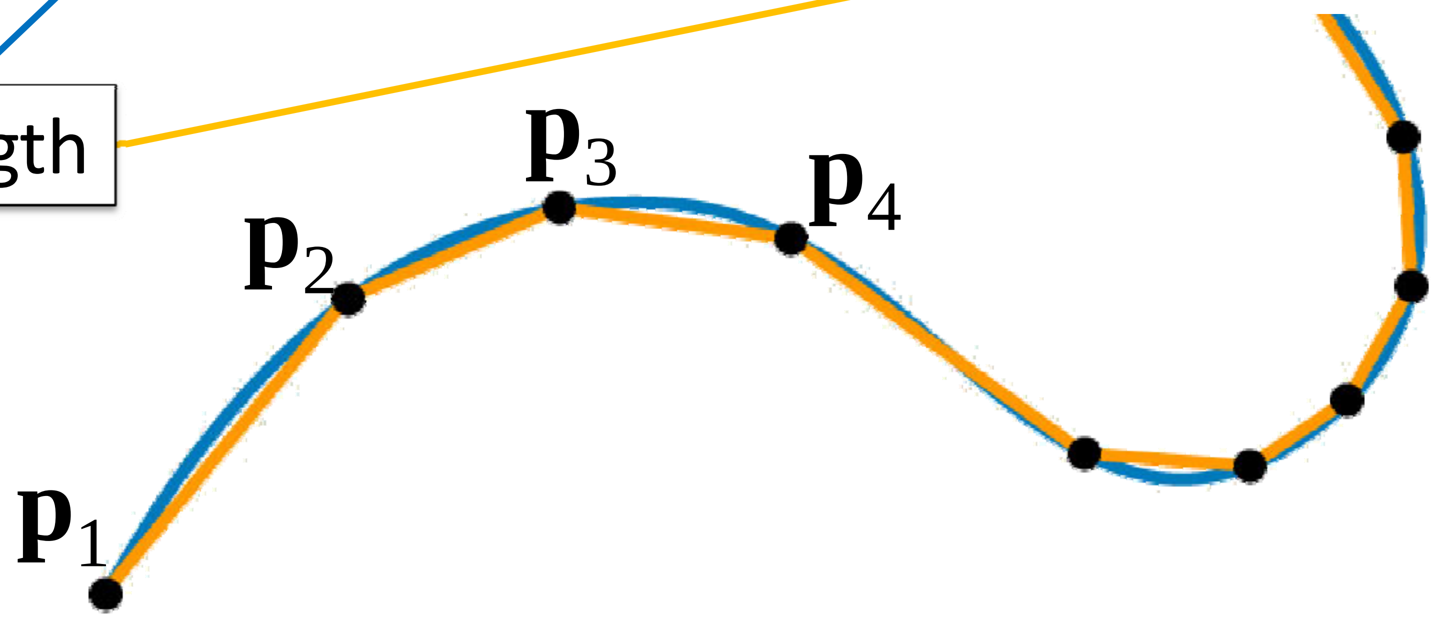
The length of a continuous curve

- Compare:

$$\boxed{\mathrm{len} = \int_{s=a}^{b} \left\| \mathbf{p}'(s) \right\| ds}$$

$$\boxed{\mathrm{len}(p) = \sum_{i=1}^{n+1} \left\| \mathbf{p}_{i+1} - \mathbf{p}_i \right\|}$$

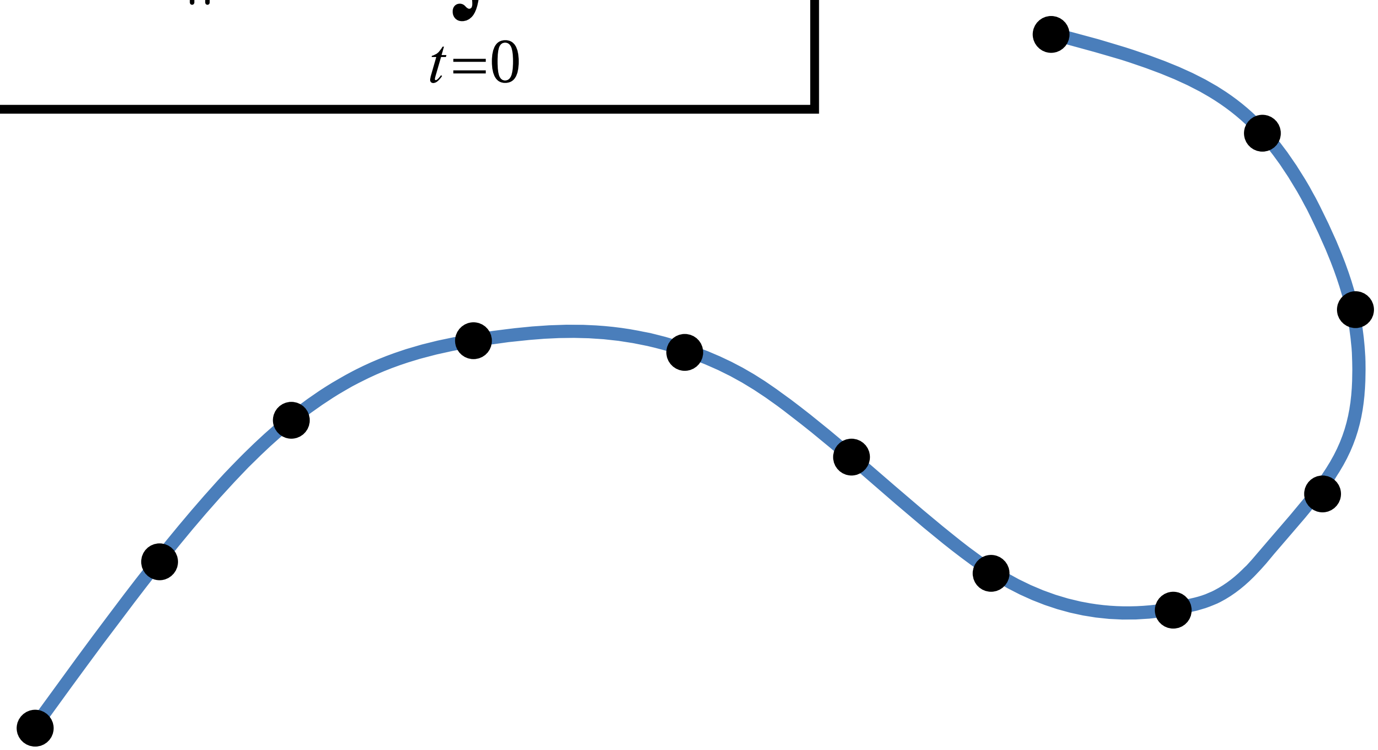
tangent length



The length of a continuous curve

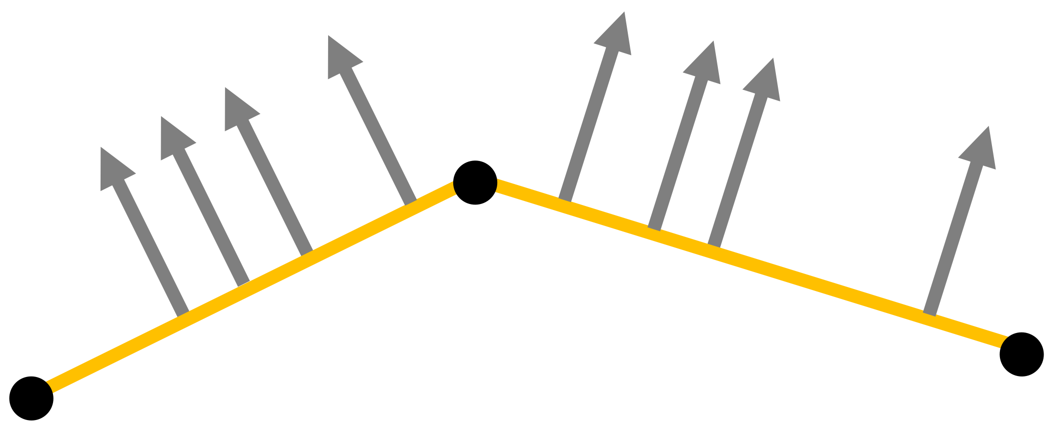
- When the parameter is arc-length:

$$\boxed{\text{len} = \int_{t=0}^{l} \|\mathbf{p}'(t)\| \, dt = \int_{t=0}^{l} 1 \, dt = l}$$



Curvature of a discrete curve

- Curvature is the change in normal direction as we travel along the curve



no change along each edge –
curvature is zero along edges

Curvature of a discrete curve

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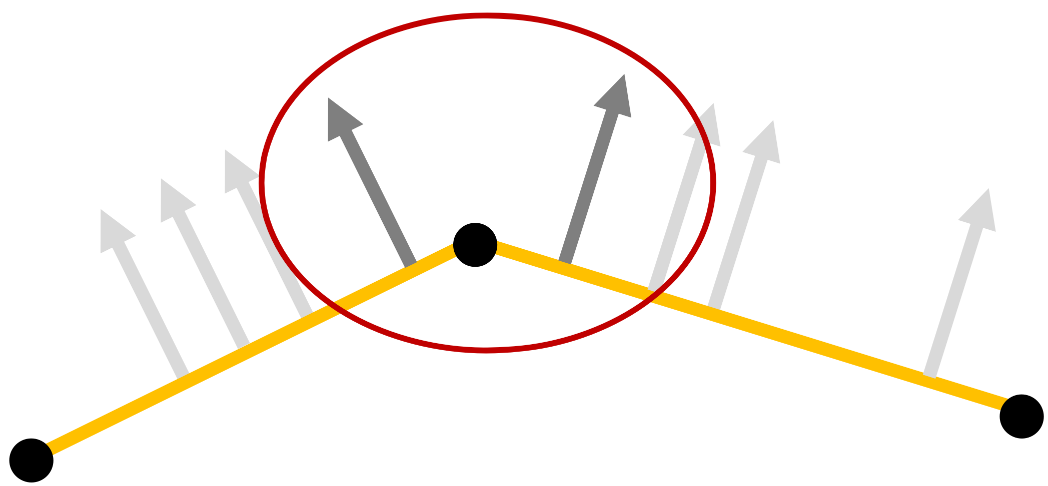

normal changes at vertices –
record the turning angle!

Curvature of a discrete curve

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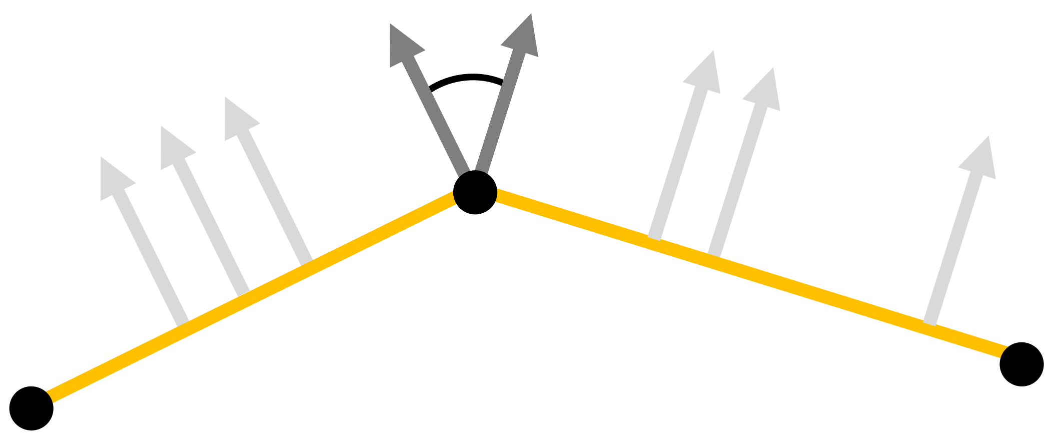

normal changes at vertices –
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Curvature of a discrete curve

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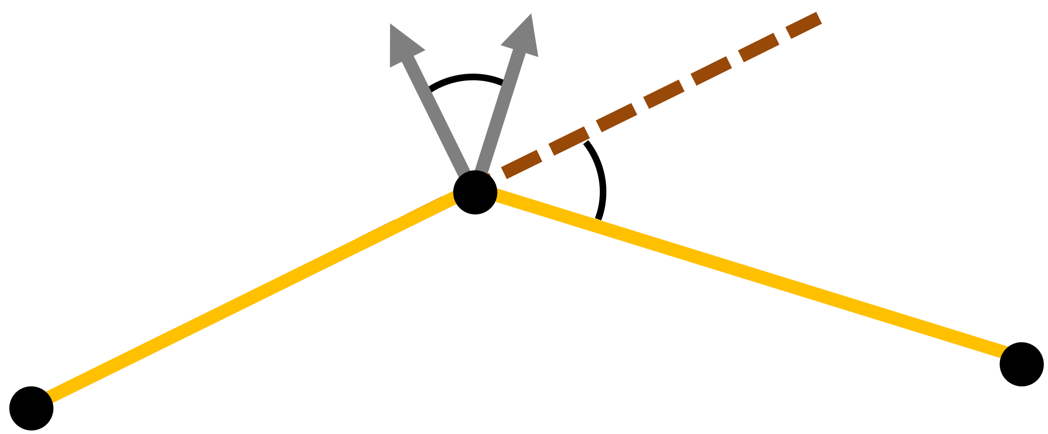
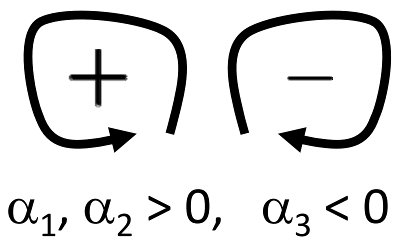
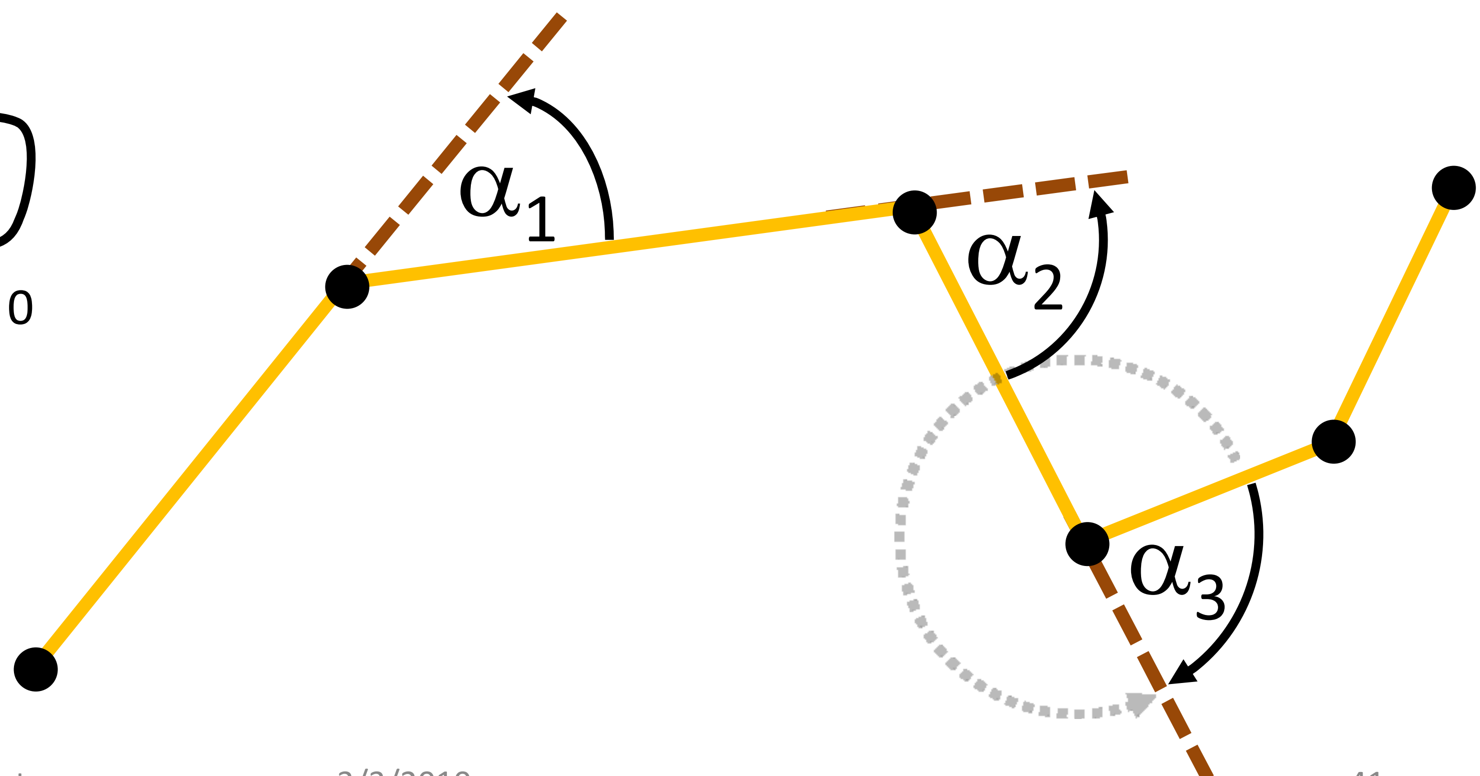


same as the turning angle
between the edges

Signed curvature of a discrete curve

- Zero along the edges
- Turning angle at the vertices
  = the change in normal direction

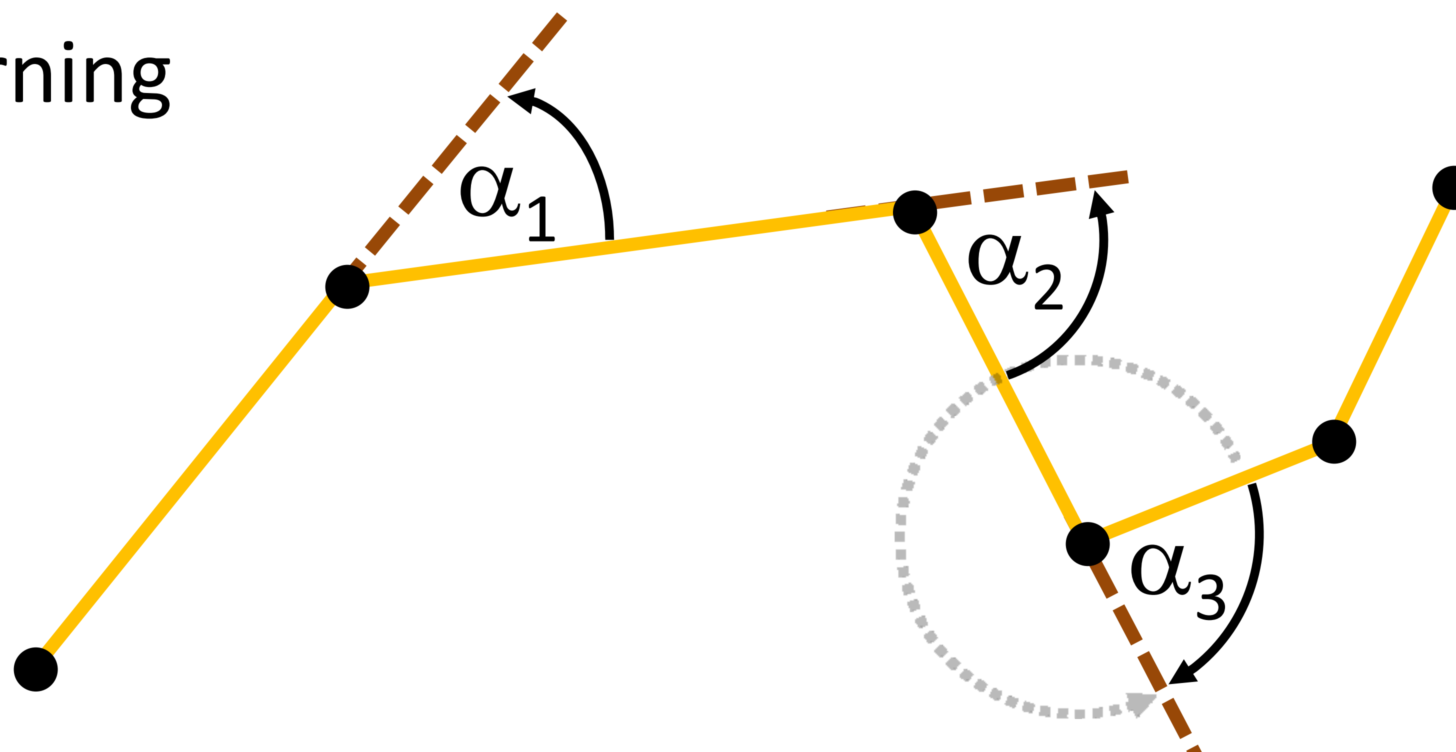




Total signed curvature

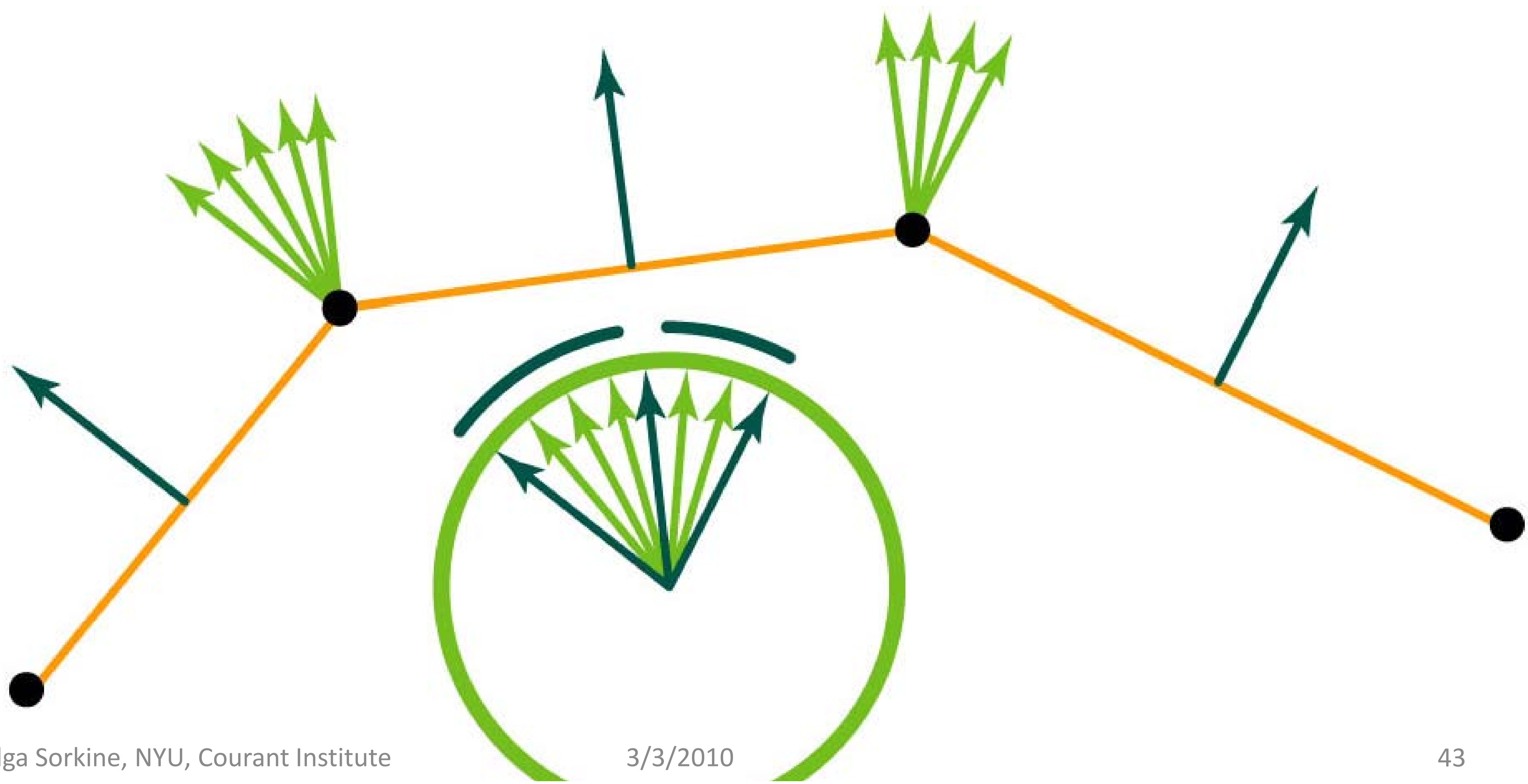
$$\boxed{\operatorname{tsc}(p) = \sum_{i=1}^{n} \alpha_i}$$

- Sum of turning angles



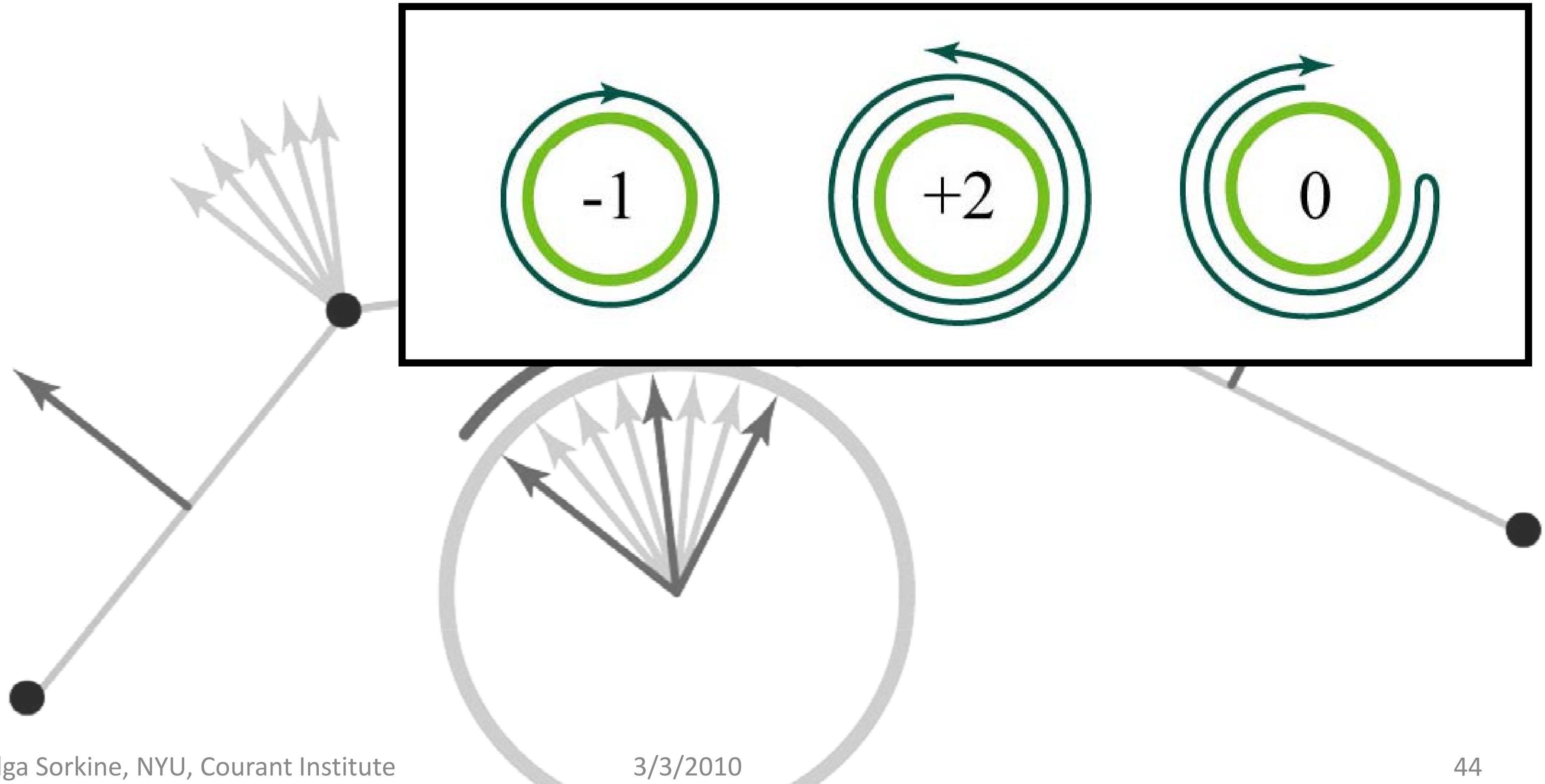
Discrete Gauss Map

- Edges map to points, vertices map to arcs.



Discrete Gauss Map

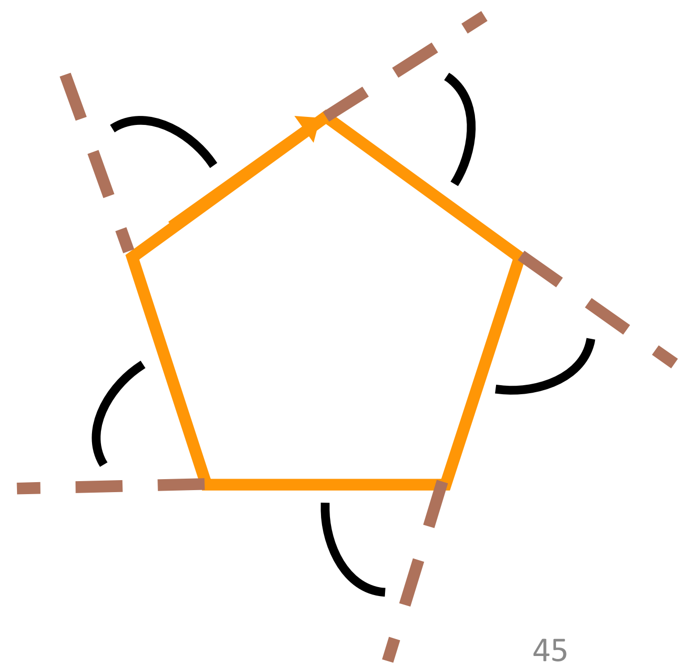
- Turning number well-defined for discrete curves.



Discrete Turning Number Theorem

$$\boxed{\operatorname{tsc}(p) = \sum_{i=1}^{n} \alpha_i = 2\pi k}$$

- For a closed curve,
  the total signed curvature is
  an integer multiple of 2π.
  - proof: sum of exterior angles



Structure preservation

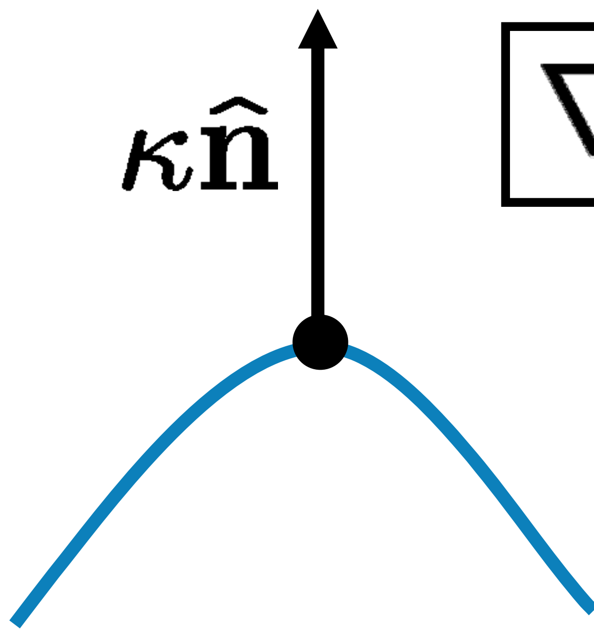
- Arbitrary discrete curve
  - total signed curvature obeys discrete turning number theorem
  - even coarse mesh (curve)
  - which continuous theorems to preserve?
    - that depends on the application…

discrete analogue of continuous theorem

Convergence

- Consider refinement sequence
  - length of inscribed polygon approaches length of smooth curve
  - in general, discrete measure approaches continuous analogue
  - which refinement sequence?
    - depends on discrete operator
    - pathological sequences may exist
  - in what sense does the operator converge? (point-wise, L_2; linear, quadratic)

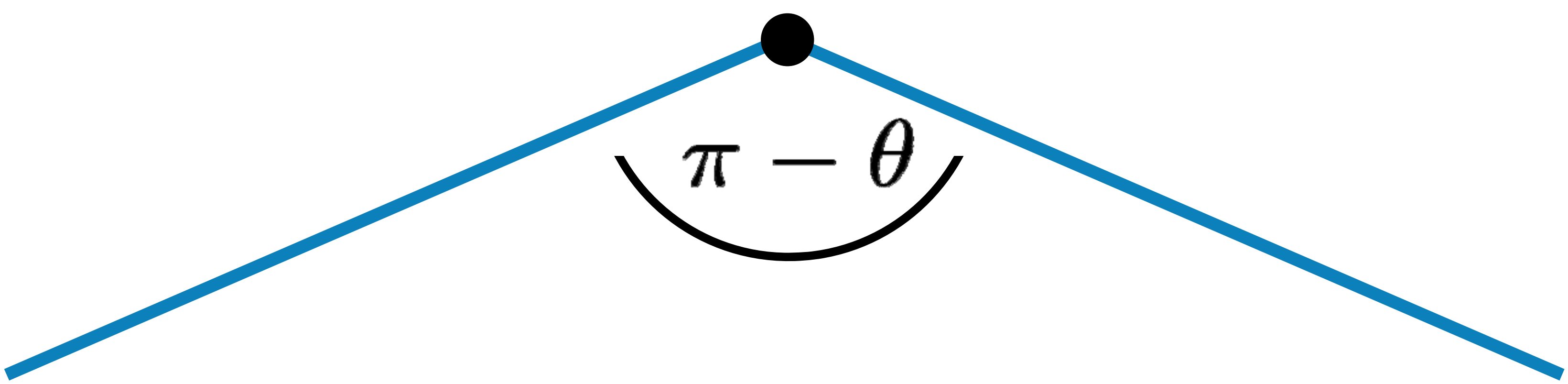
Curvature normal = length gradient



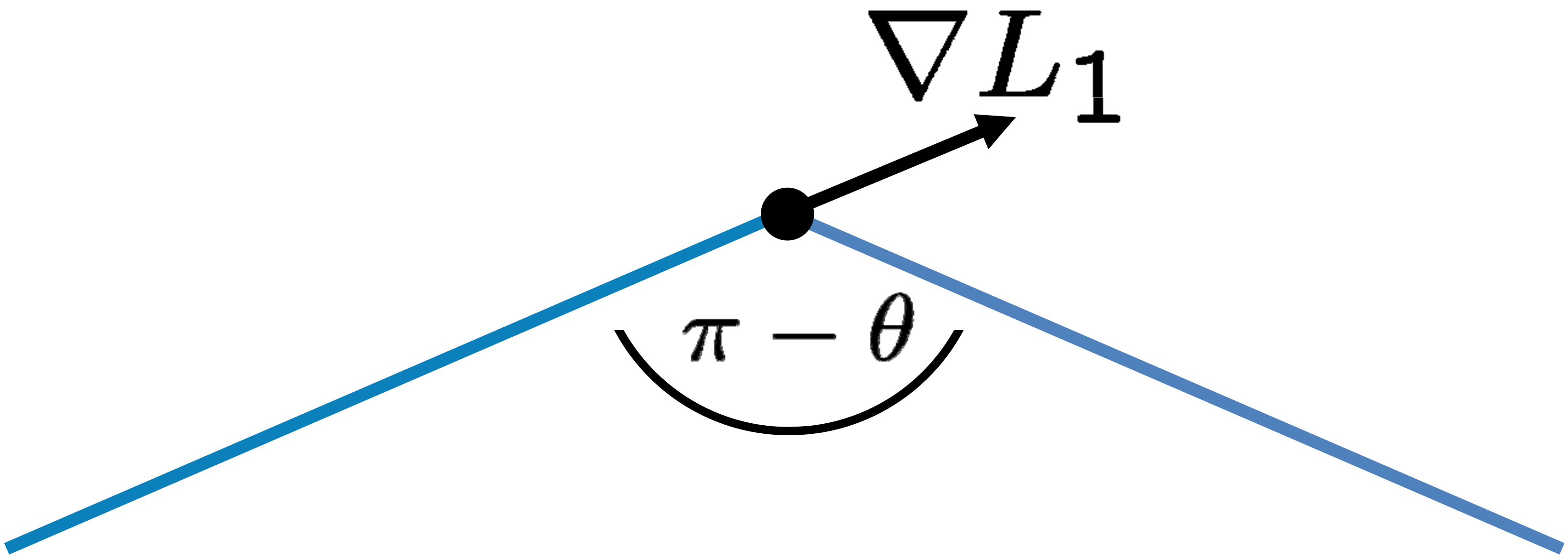
$$\boxed{\nabla L = \kappa\hat{\mathbf{n}}}$$

- Can use this to define discrete curvature!

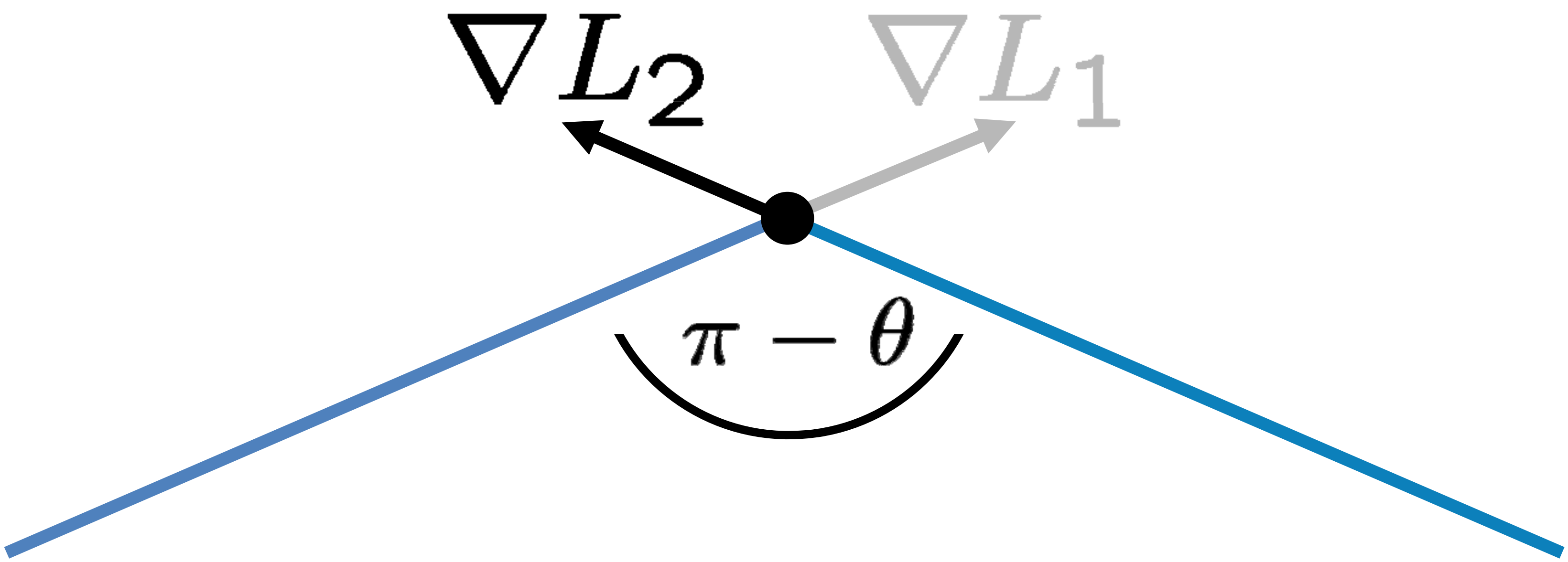
Curvature normal = length gradient



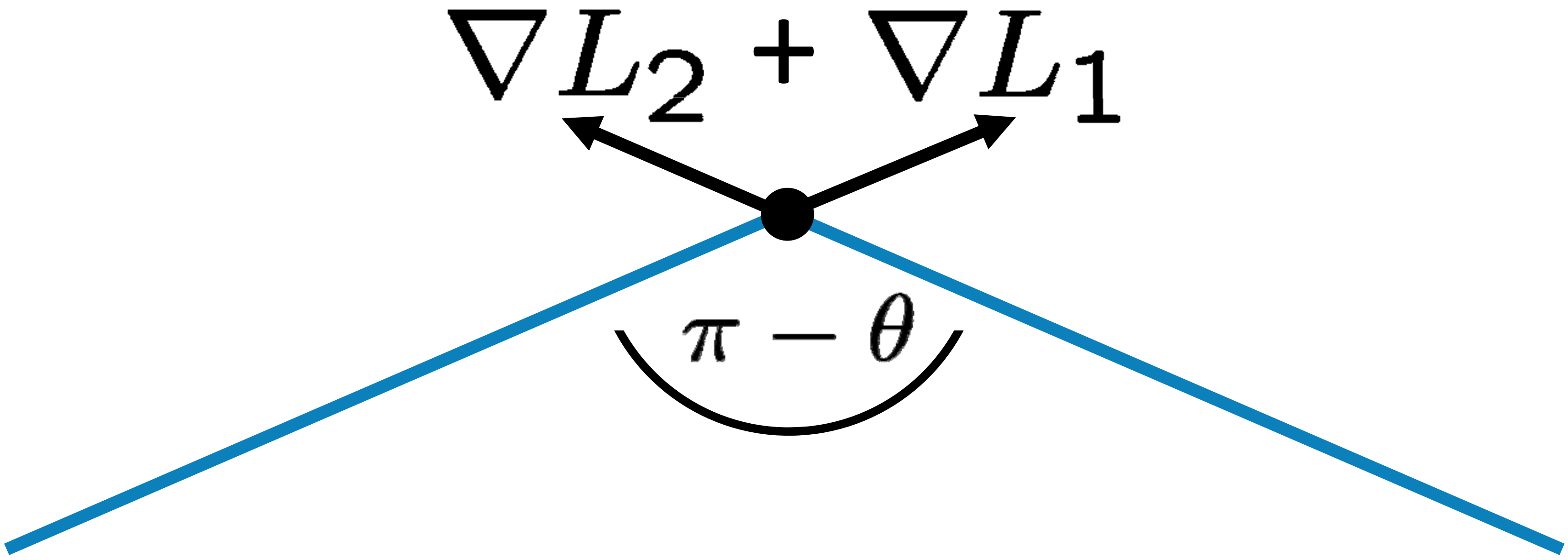
Curvature normal = length gradient



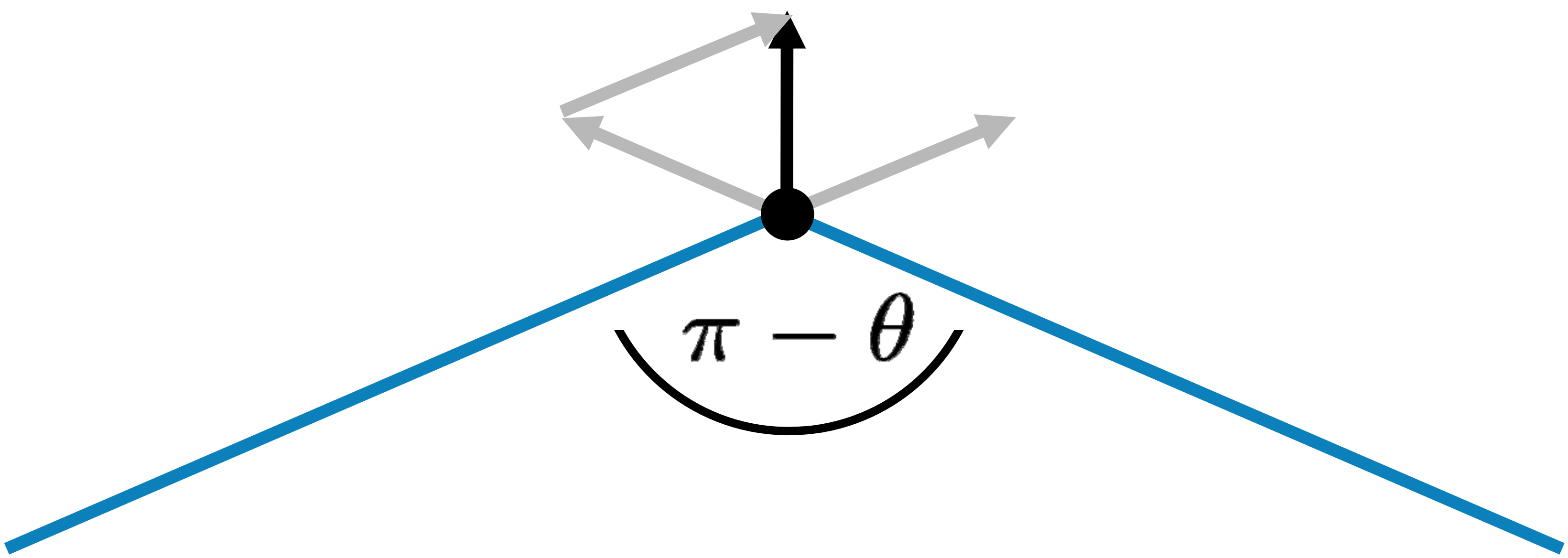
Curvature normal = length gradient



Curvature normal = length gradient

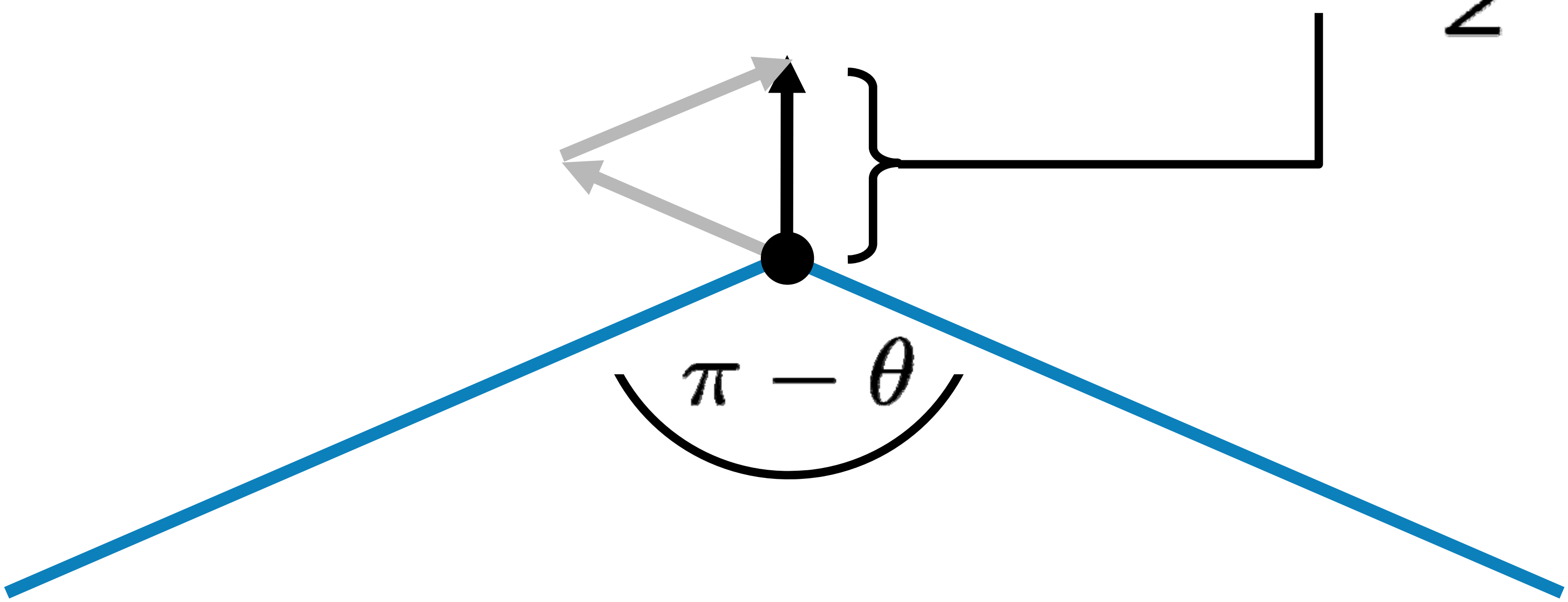


Curvature normal = length gradient



Curvature normal = length gradient

$$\nabla L = \kappa \hat{\mathbf{n}} = 2 \sin \frac{\theta}{2} \hat{\mathbf{n}}$$



Recap

Structure-preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.