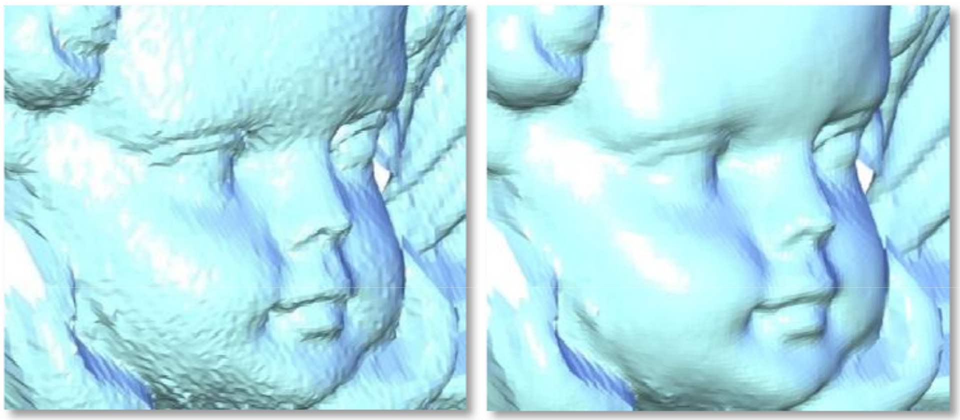
G22.3033-008, Spring 2010

Geometric Modeling

Differential Geometry of Surfaces

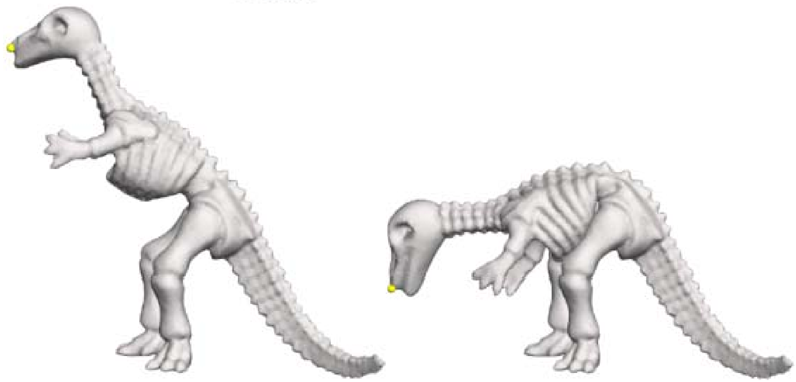
Motivation

- Smoothness
  - Mesh smoothing



- Adaptive tessellation
  - Mesh decimation

- Shape preserving mesh editing



Surfaces

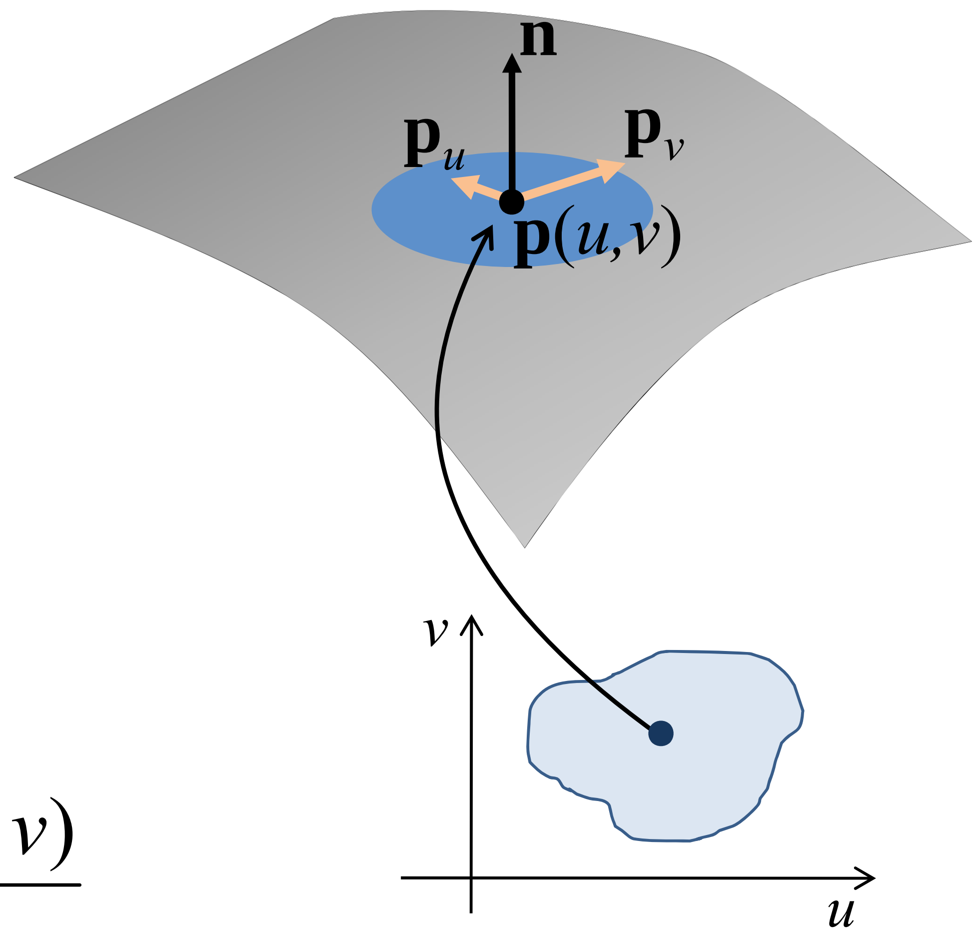
Parametric form

- Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \Re^2$$

- Tangent plane at point $\mathbf{p}(u,v)$ is spanned by

$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \qquad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$



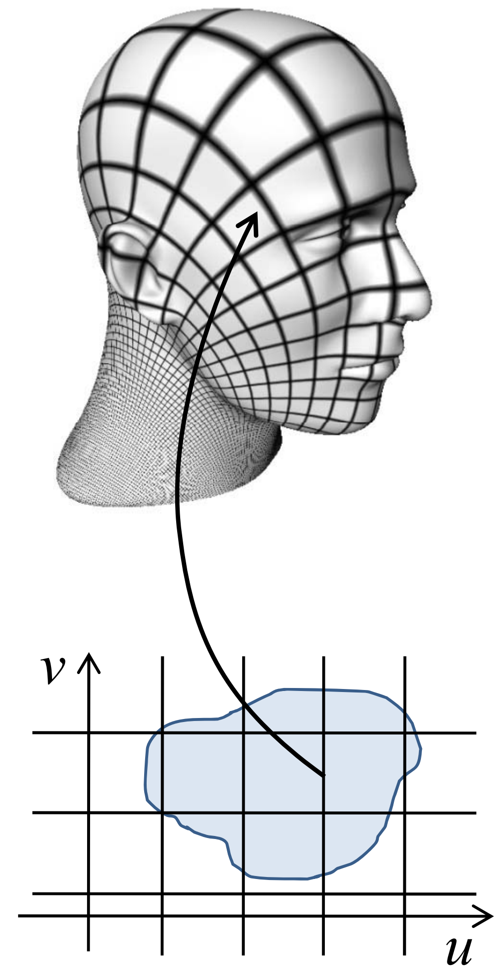
Surfaces

Isoparametric lines

- Lines on the surface when keeping one parameter fixed

$$\gamma_{u_0}(v) = \mathbf{p}(u_0, v)$$

$$\gamma_{v_0}(u) = \mathbf{p}(u, v_0)$$



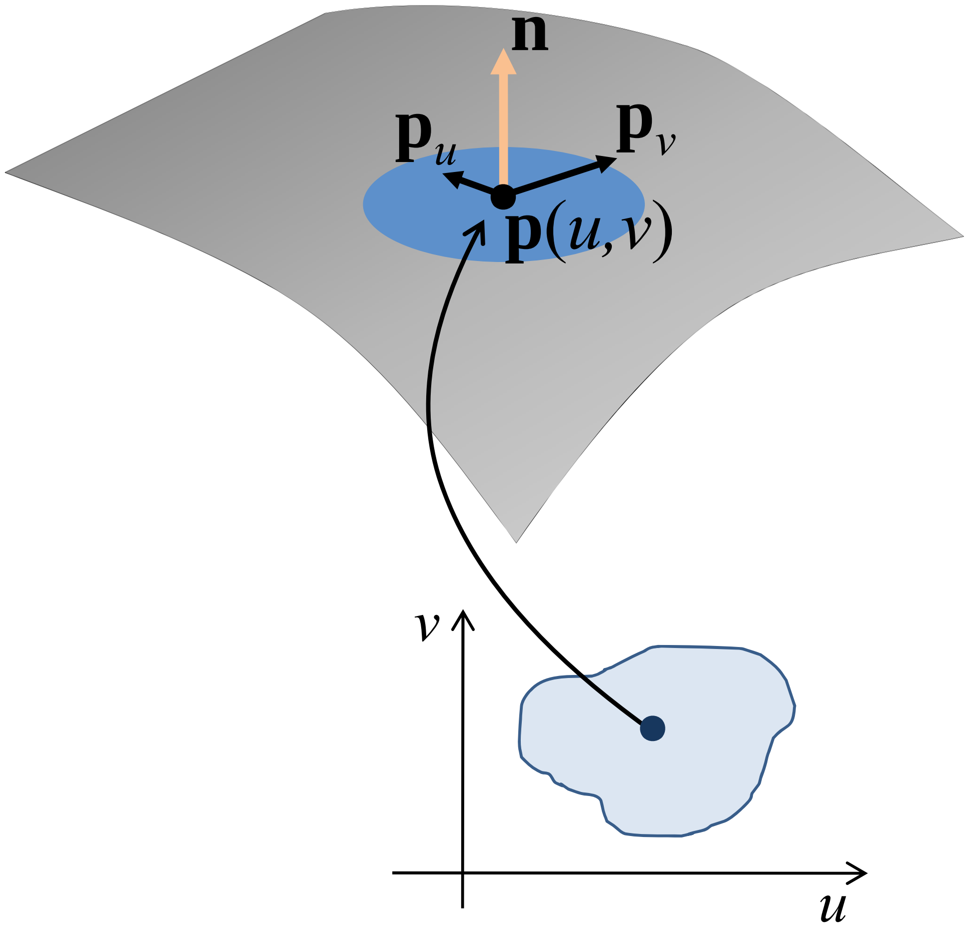
Surfaces

- Surface normal:

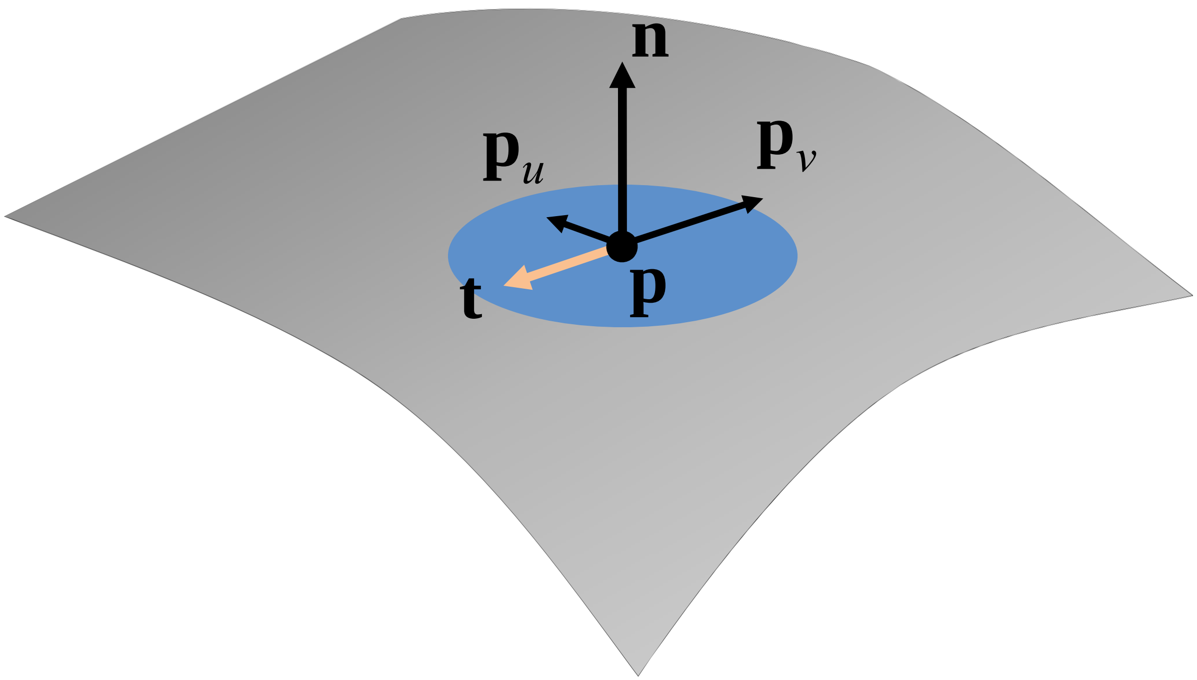
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\left\| \mathbf{p}_u \times \mathbf{p}_v \right\|}$$

  - Assuming *regular* parameterization, i.e.,

$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$



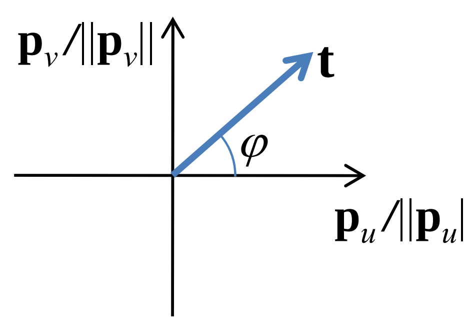
Normal curvature



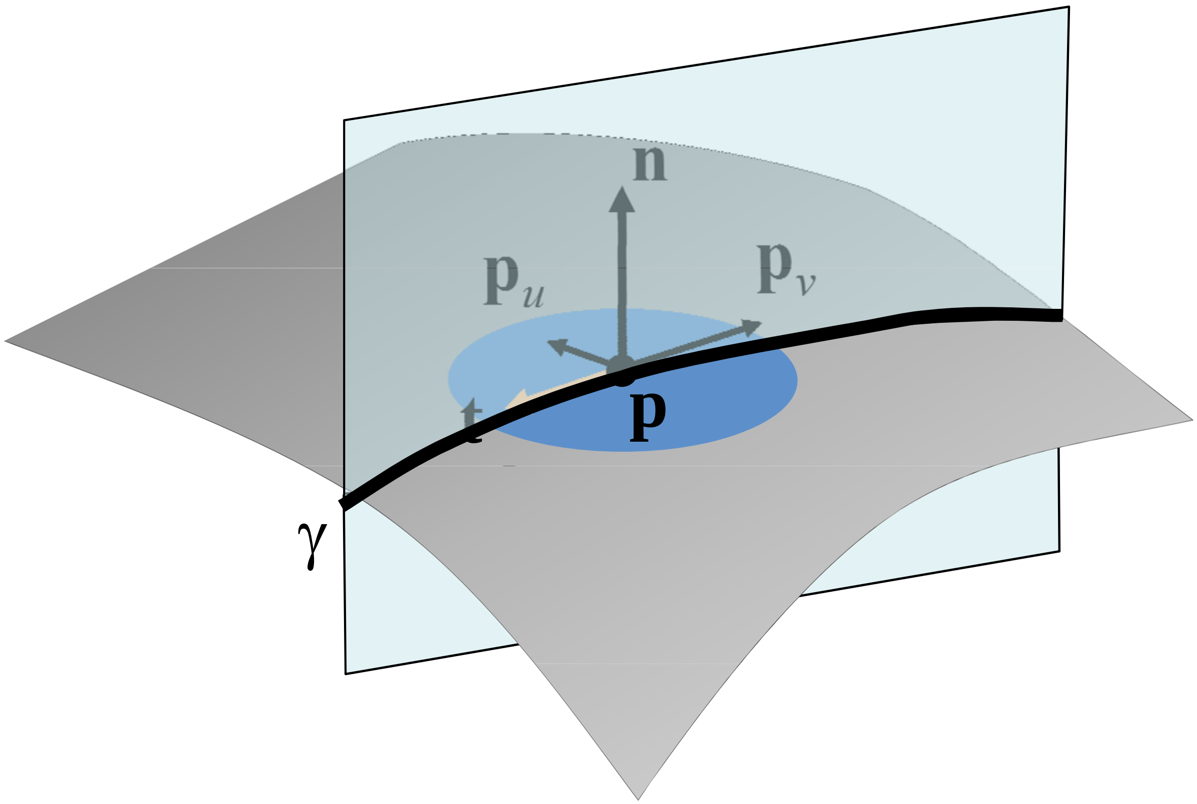
$$\mathbf{n} = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\left\| \mathbf{p}_u \times \mathbf{p}_v \right\|}$$

Direction **t** in the tangent plane:

$$\mathbf{t} = \cos\varphi \frac{\mathbf{p}_u}{\left\|\mathbf{p}_u\right\|} + \sin\varphi \frac{\mathbf{p}_v}{\left\|\mathbf{p}_v\right\|}$$



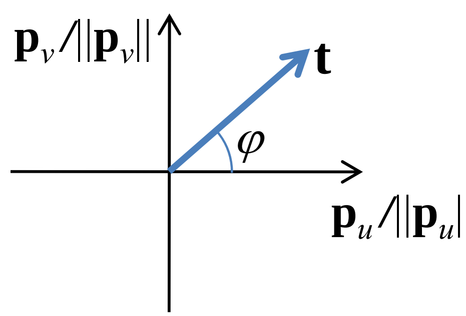
Normal curvature



The curve γ is the intersection of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$.

Normal curvature:

$$\kappa_n(\varphi) = \kappa(\gamma(\mathbf{p}))$$



Surface curvatures

- Principal curvatures
  - Maximal curvature $\kappa_1 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$
  - Minimal curvature $\kappa_2 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$
- Mean curvature

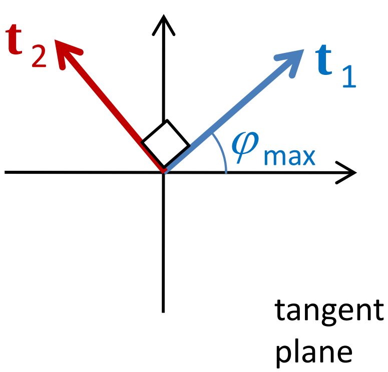
$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi)\, \mathrm{d}\varphi$$

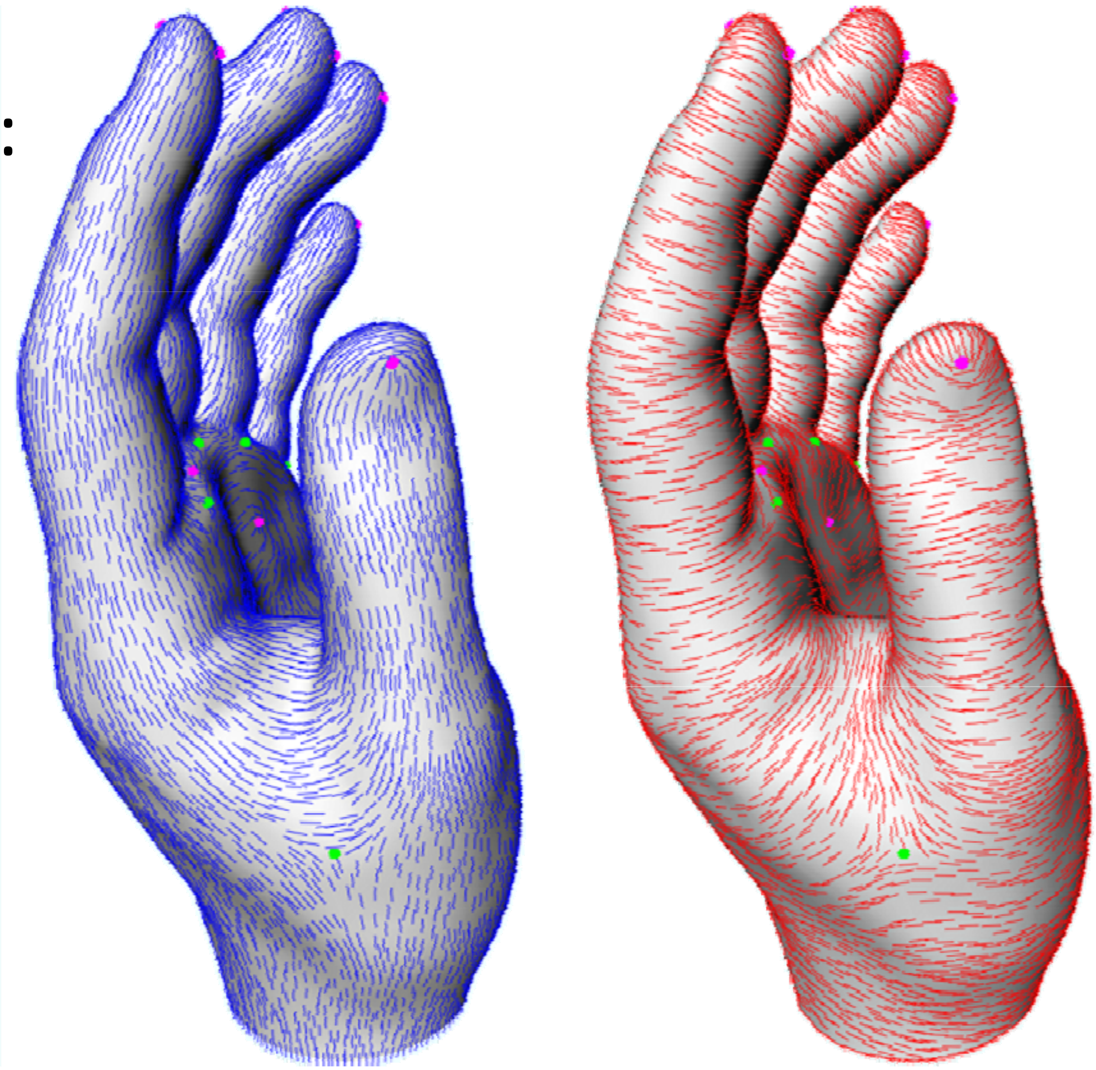
- Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2$$

Principal directions

- Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}



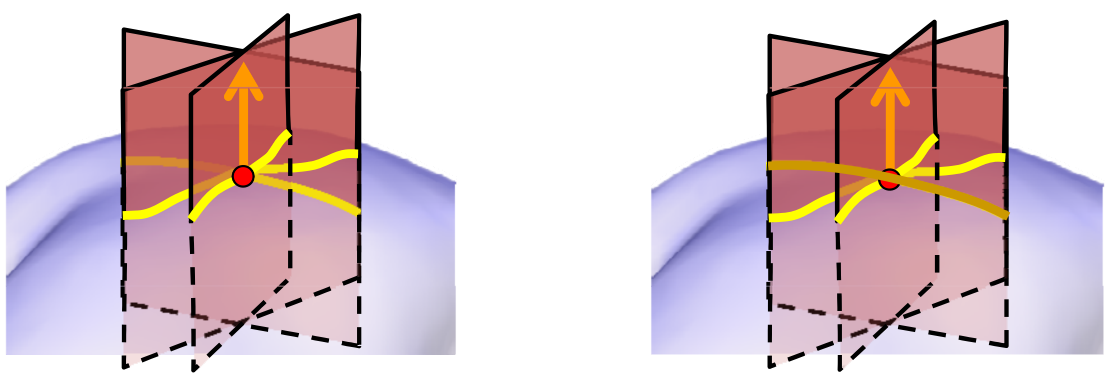


min curvature max curvature

Mean curvature

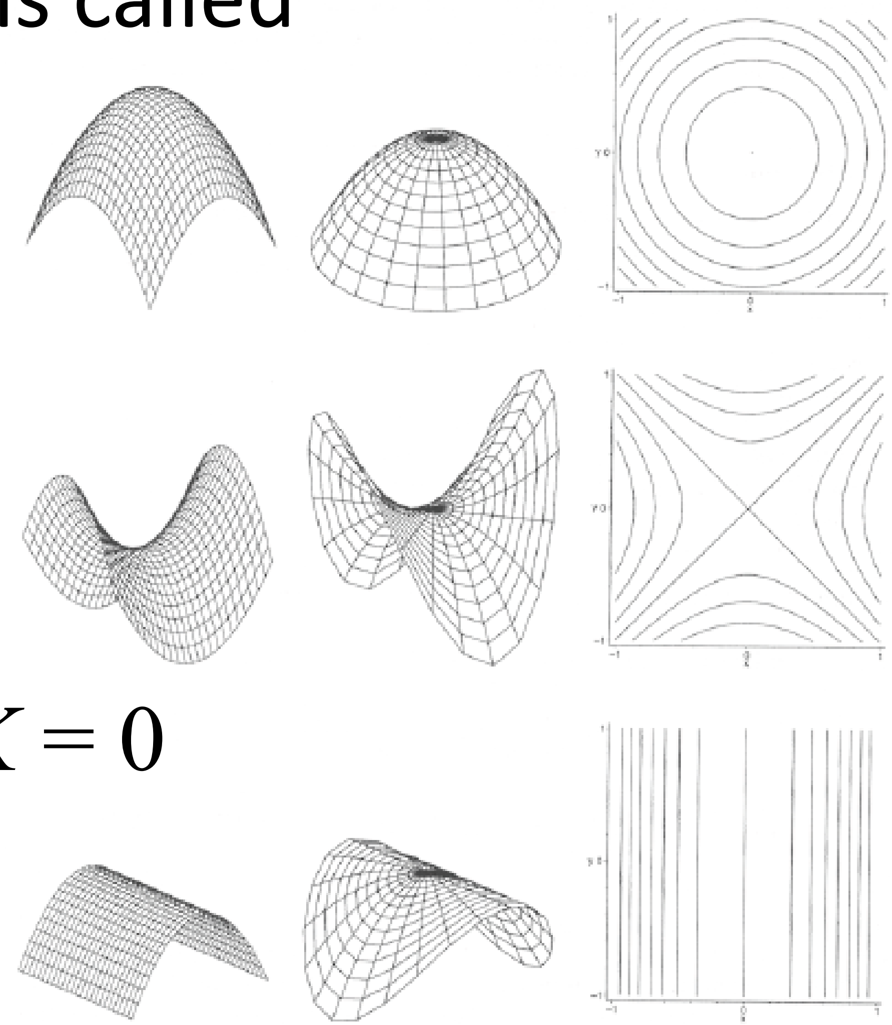
- Intuition for mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\varphi)\,\mathrm{d}\varphi$$



Classification

- A point $\mathbf{p}$ on the surface is called
  - Elliptic, if $K > 0$
  - Parabolic, if $K = 0$
  - Hyperbolic, if $K < 0$
  - Umbilical, if $\kappa_1 = \kappa_2$
- Developable surface $\Leftrightarrow K = 0$

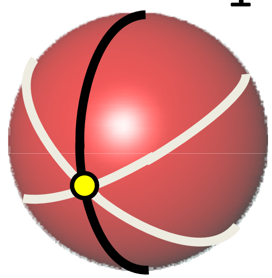


Classification

Local surface shape by curvatures

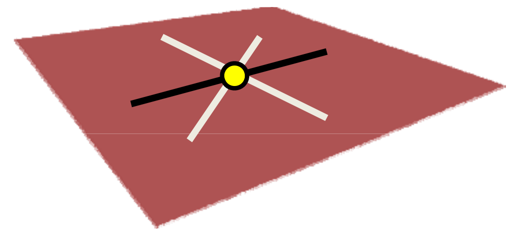
Isotropic:
all directions are principal directions

$K > 0, \kappa_1 = \kappa_2$



spherical (umbilical)

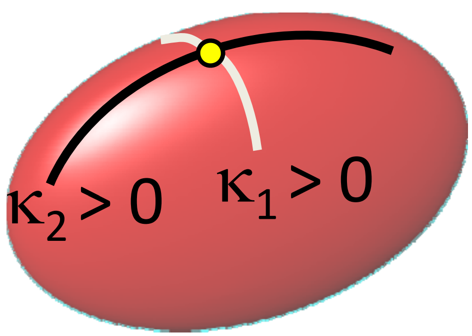
$K = 0$



planar

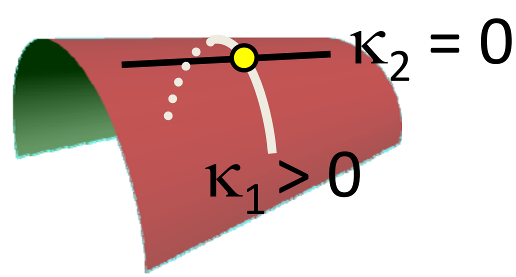
Anisotropic:
2 distinct principal directions

$K > 0$



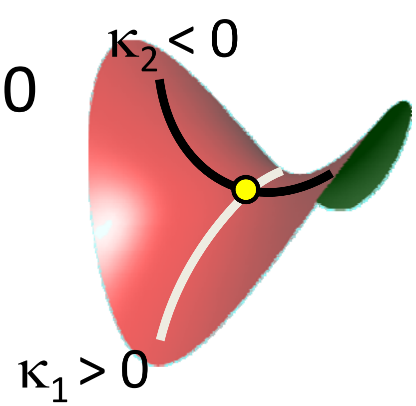
elliptic

$K = 0$



parabolic

$K < 0$



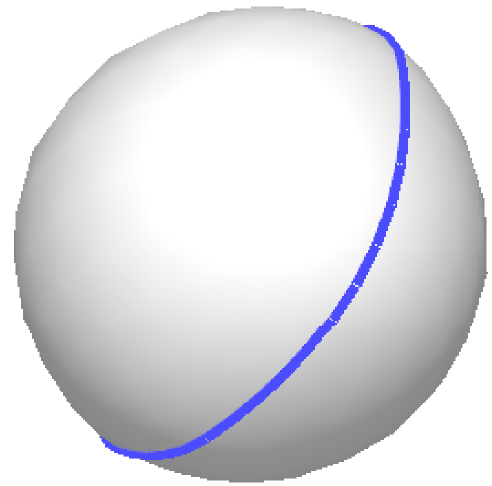
hyperbolic

Measuring surface smoothness

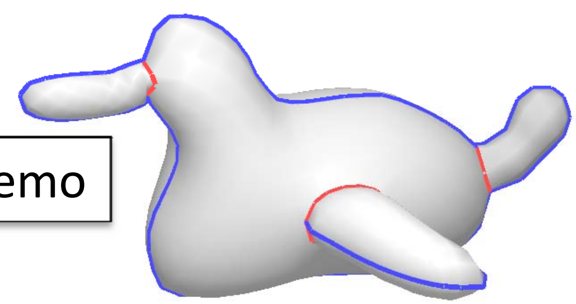
Example – curvature variation

$$\int_M \left(\frac{\partial \kappa_1}{\partial \mathbf{t}_1}\right)^2 + \left(\frac{\partial \kappa_2}{\partial \mathbf{t}_2}\right)^2 \mathrm{d}A$$

Spheres minimize curvature variation



FiberMesh demo



Gauss-Bonnet Theorem

- For a closed surface M:

$$\int_M K \, \mathrm{d}A = 2\pi \chi(M)$$

- Compare with planar curves:

$$\int_\gamma \kappa \, \mathrm{d}s = 2\pi k$$

Laplace operator

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \operatorname{div} \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \ldots$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami operator

- Extension of Laplace to functions on manifolds

Laplace-
Beltrami

gradient
operator

$$\Delta_M f = \mathrm{div}_M \nabla_M f$$

function on
surface M

divergence
operator

Laplace-Beltrami operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_M \mathbf{p} = \mathrm{div}_M \nabla_M \mathbf{p} = -2H\mathbf{n}$$

surface coordinates

divergence operator

surface normal

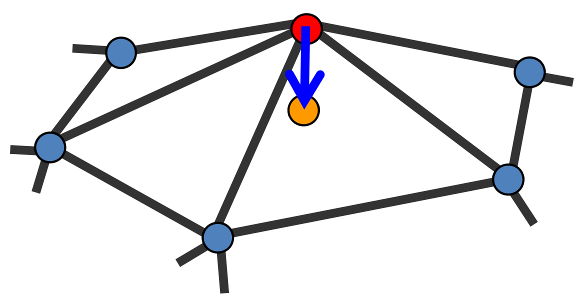
Differential geometry on meshes

- Assumption: meshes are piecewise linear approximations of smooth surfaces

- Can try fitting a smooth surface locally (say, a polynomial) and find differential quantities analytically
- But: it is often too slow for interactive setting and error prone

Discrete differential operators

- Approach: approximate differential properties at point $\mathbf{v}$ as spatial average over local mesh neighborhood $N(\mathbf{v})$ where typically
  - $\mathbf{v}$ = mesh vertex
  - $N_k(\mathbf{v})$ = k-ring neighborhood or local geodesic ball



$$\boxed{\Delta_M \mathbf{p} = -2H\mathbf{n}}$$

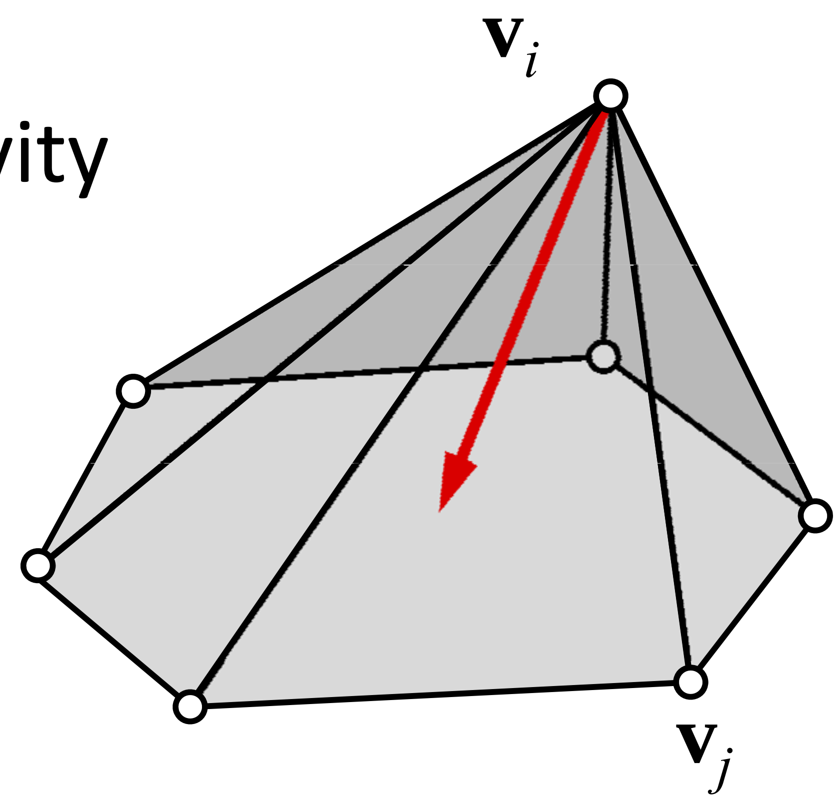
Discrete Laplace-Beltrami

- Uniform discretization: $L(\mathbf{v})$ or $\Delta\mathbf{v}$

$$L_u(\mathbf{v}_i) = \frac{1}{|N_1(\mathbf{v}_i)|} \sum_{\mathbf{v}_j \in N_1(\mathbf{v}_i)} (\mathbf{v}_j - \mathbf{v}_i) = \frac{1}{d_i}\left(\left(\sum_{\mathbf{v}_j \in N_1(\mathbf{v}_i)} \mathbf{v}_j\right) - d_i \mathbf{v}_i\right)$$

- Depends only on connectivity = simple and efficient

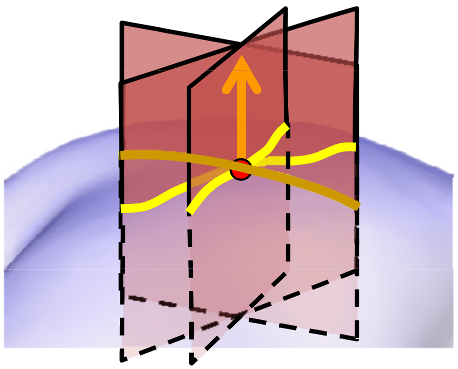
- Bad approximation for irregular triangulations

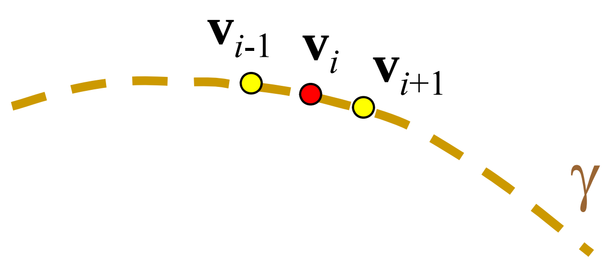


$$\boxed{\Delta_M \mathbf{p} = -2H\mathbf{n}}$$

Discrete Laplace-Beltrami

- Intuition for uniform discretization





$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)\,\mathrm{d}\varphi$$

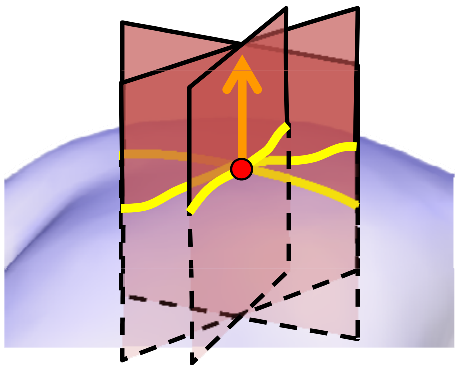
$$\kappa\mathbf{n} = \gamma''$$

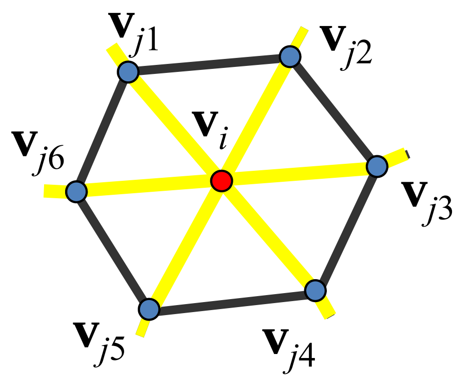
$$\gamma'' \approx \frac{1}{t}\big((\mathbf{v}_i - \mathbf{v}_{i-1}) - (\mathbf{v}_{i+1} - \mathbf{v}_i)\big) = -\frac{1}{t}(\mathbf{v}_{i-1} + \mathbf{v}_{i+1} - 2\mathbf{v}_i)$$

$$\boxed{\Delta_M \mathbf{p} = -2H\mathbf{n}}$$

Discrete Laplace-Beltrami

- Intuition for uniform discretization





$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)\,\mathrm{d}\varphi$$

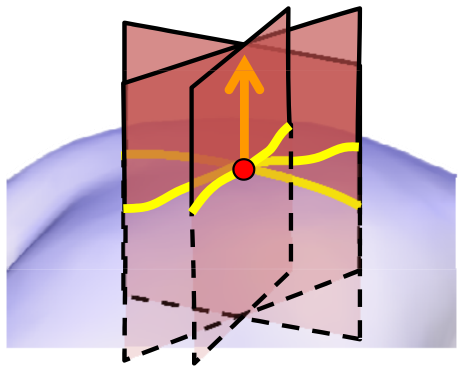
$$\begin{aligned} &\mathbf{v}_{j1} + \mathbf{v}_{j4} - 2\mathbf{v}_i \quad + \\ &\mathbf{v}_{j2} + \mathbf{v}_{j5} - 2\mathbf{v}_i \quad + \\ &\mathbf{v}_{j3} + \mathbf{v}_{j6} - 2\mathbf{v}_i \quad = \end{aligned}$$

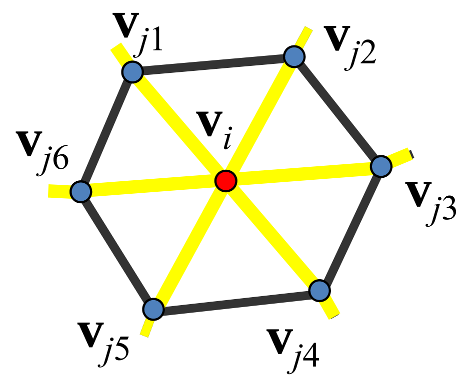
$$6\,L(\mathbf{v}_i) = \sum_{k=1}^{6} \mathbf{v}_{jk} - 6\mathbf{v}_i \;\approx\; -6\cdot 2H\mathbf{n}$$

$$\boxed{\Delta_M \mathbf{p} = -2H\mathbf{n}}$$

Discrete Laplace-Beltrami

- Intuition for uniform discretization





$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\varphi)\,\mathrm{d}\varphi$$

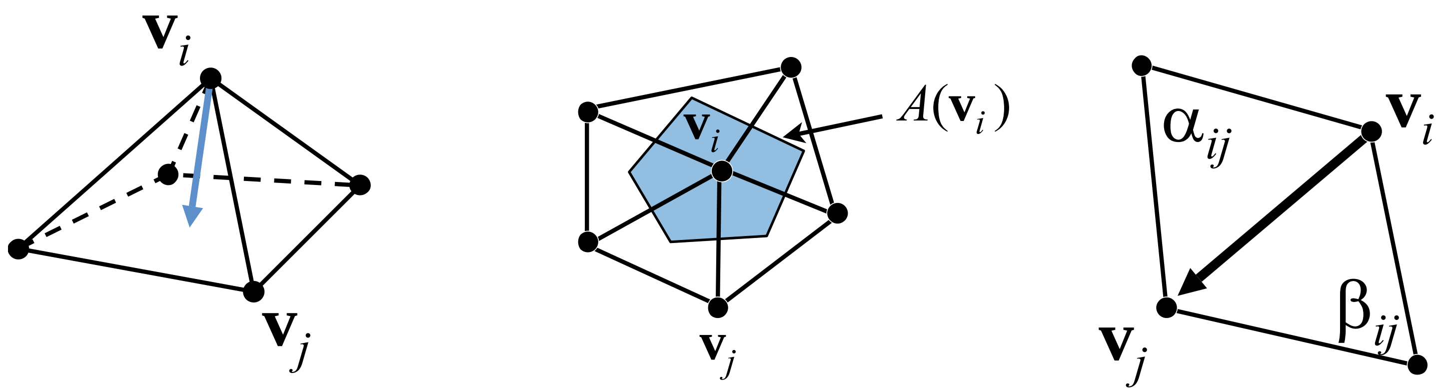
$$\begin{aligned}
&\mathbf{v}_{j1} + \mathbf{v}_{j4} - 2\mathbf{v}_i \quad + \\
&\mathbf{v}_{j2} + \mathbf{v}_{j5} - 2\mathbf{v}_i \quad + \\
&\mathbf{v}_{j3} + \mathbf{v}_{j6} - 2\mathbf{v}_i \quad =
\end{aligned}$$

$$\boxed{L(\mathbf{v}_i) = \frac{1}{6}\left(\sum_{k=1}^{6} \mathbf{v}_{jk} - 6\mathbf{v}_i\right) \approx -2H\mathbf{n}}$$

Discrete Laplace-Beltrami

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{2A(\mathbf{v}_i)} \sum_{\mathbf{v}_j \in N_1(\mathbf{v}_i)} \left(\cot \alpha_{ij} + \cot \beta_{ij}\right)\left(\mathbf{v}_j - \mathbf{v}_i\right)$$

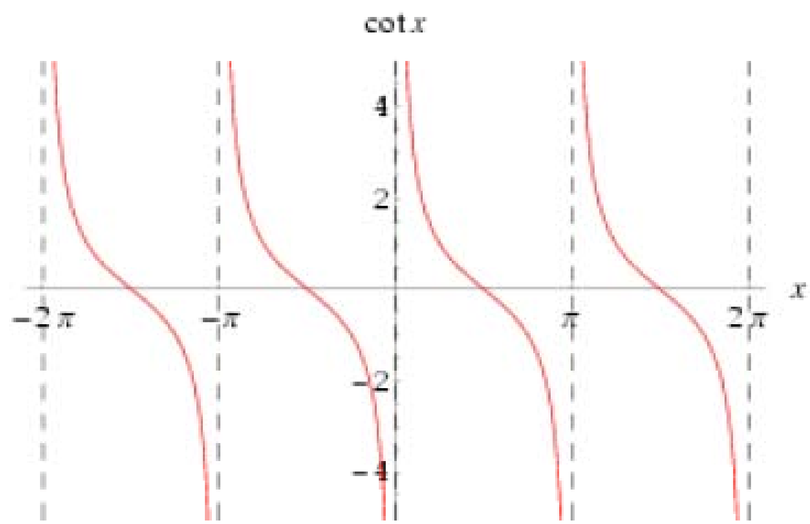


Discrete Laplace-Beltrami

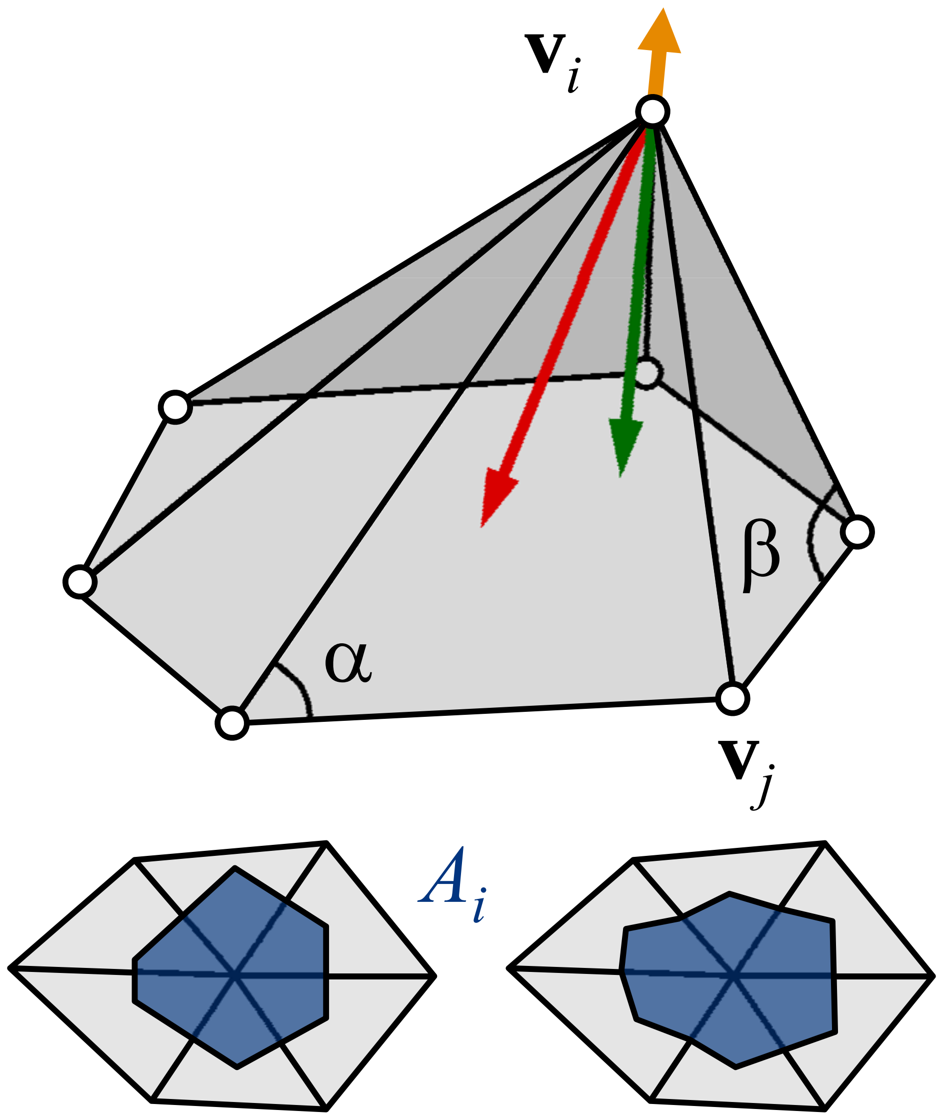
- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{2A(\mathbf{v}_i)} \sum_{\mathbf{v}_j \in N_1(\mathbf{v}_i)} \left(\cot\alpha_{ij} + \cot\beta_{ij}\right)\left(\mathbf{v}_j - \mathbf{v}_i\right)$$

- Problems
  - Potentially negative weights
  - Depends on geometry

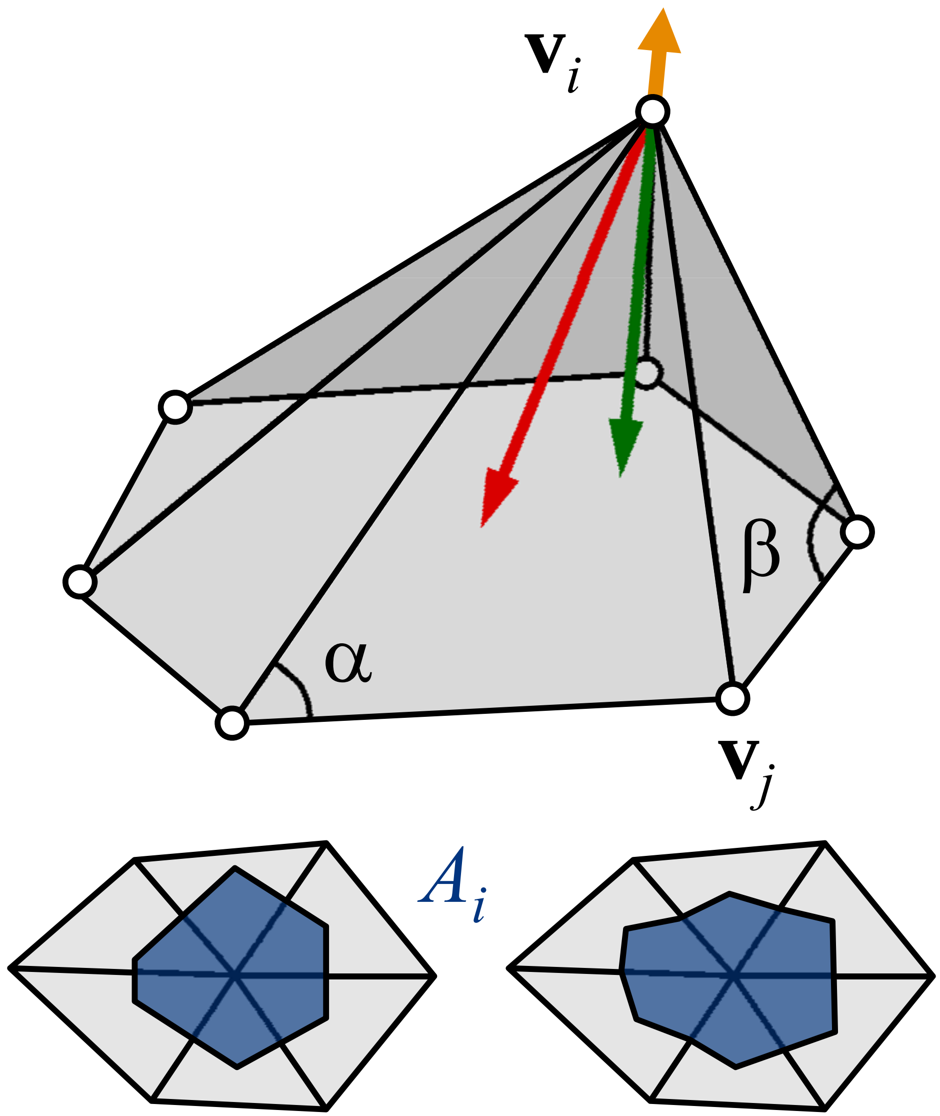


Discrete Laplace-Beltrami



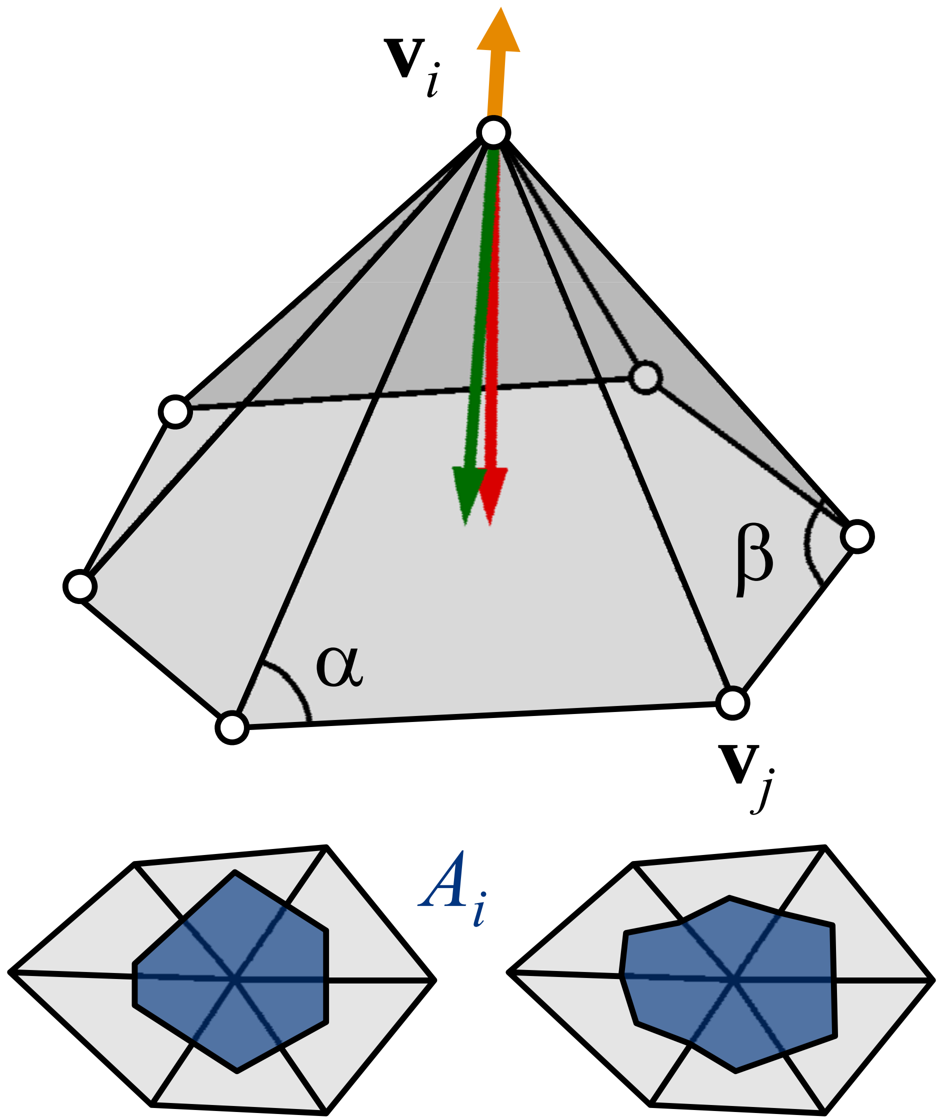
- Laplacian operators
  - **Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$**
  - **Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$**
  - **Mean curvature normal**

Discrete Laplace-Beltrami



- Laplacian operators
  - **Normalized Uniform Laplacian**
  - **Normalized Cotangent Laplacian**
  - **Mean curvature normal**
- **Cotangent Laplacian** = **mean curvature normal** x **vertex area** (**A_i**)
- **For nearly equal edge lengths**
  Uniform ≈ **Cotangent**

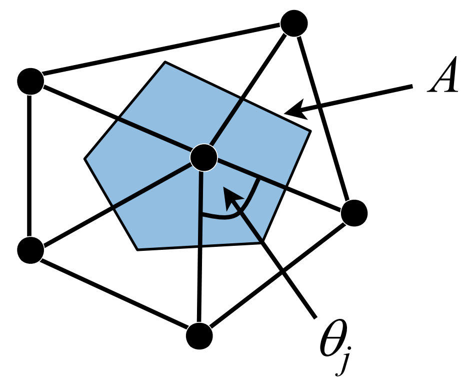
Discrete Laplace-Beltrami



- Laplacian operators
  - Normalized Uniform Laplacian
  - Normalized Cotangent Laplacian
  - Mean curvature normal
- Cotangent Laplacian = mean curvature normal x vertex area (A_i)
- **For nearly equal edge lengths**
  Uniform ≈ Cotangent

Discrete curvatures

- Mean curvature (sign defined according to normal) $H(\mathbf{v}_i) = \left\| L_c(\mathbf{v}_i) \right\|$



- Gaussian curvature

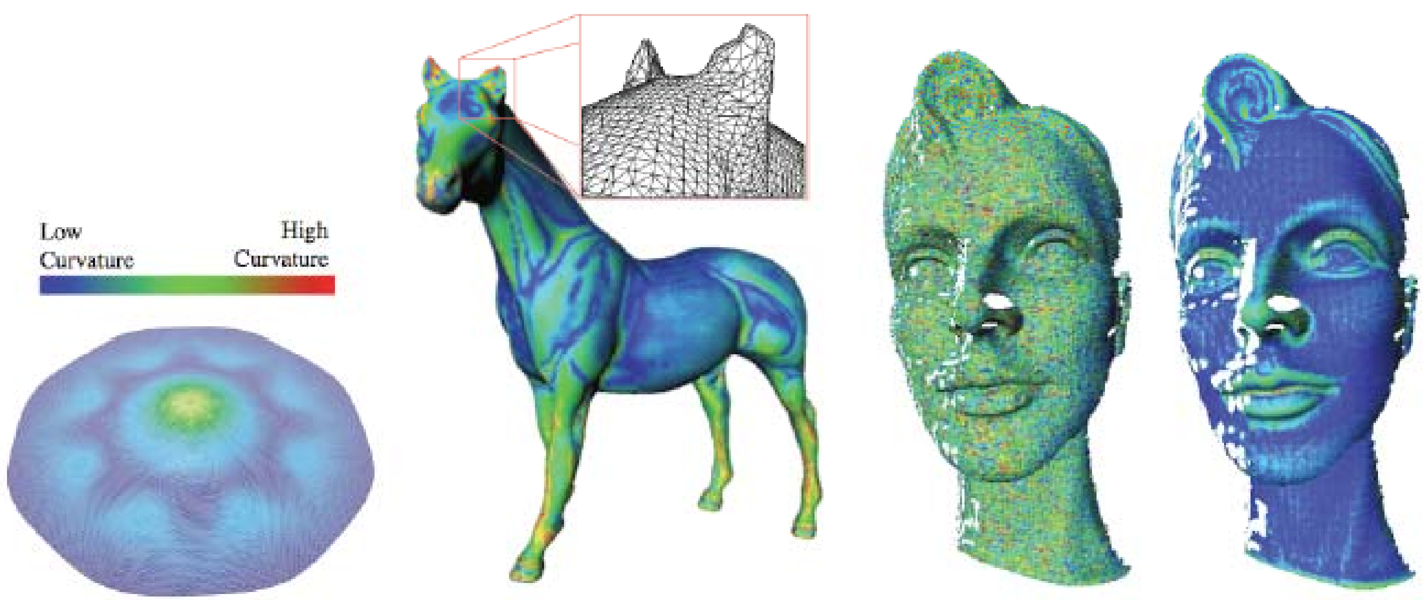
$$G(\mathbf{v}_i) = (2\pi - \sum_j \theta_j) / A(\mathbf{v}_i)$$

- Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - G} \qquad \kappa_2 = H - \sqrt{H^2 - G}$$

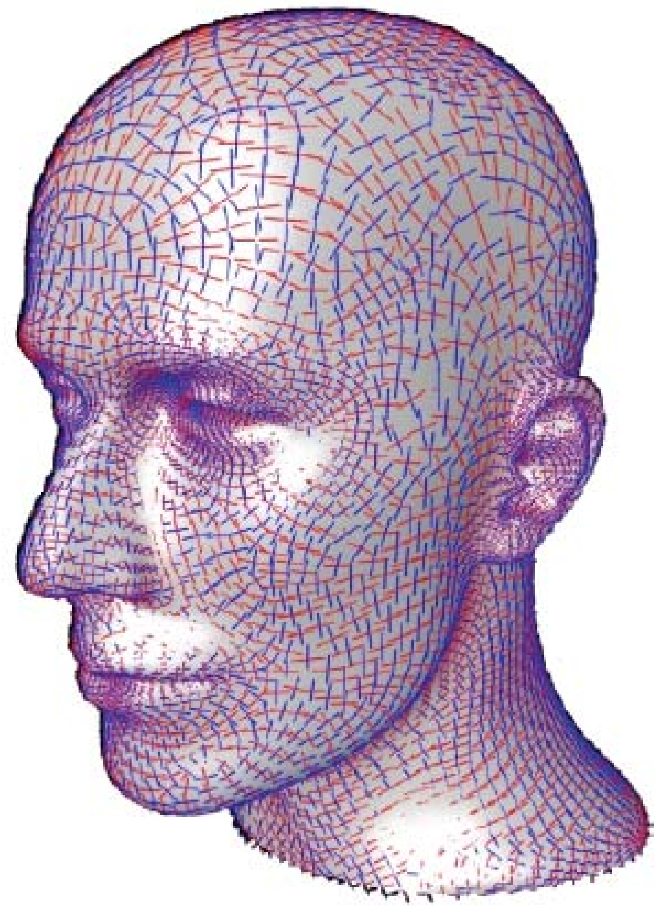
Links and literature

- M. Meyer, M. Desbrun, P. Schroeder, A. Barr *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, VisMath, 2002



Links and literature

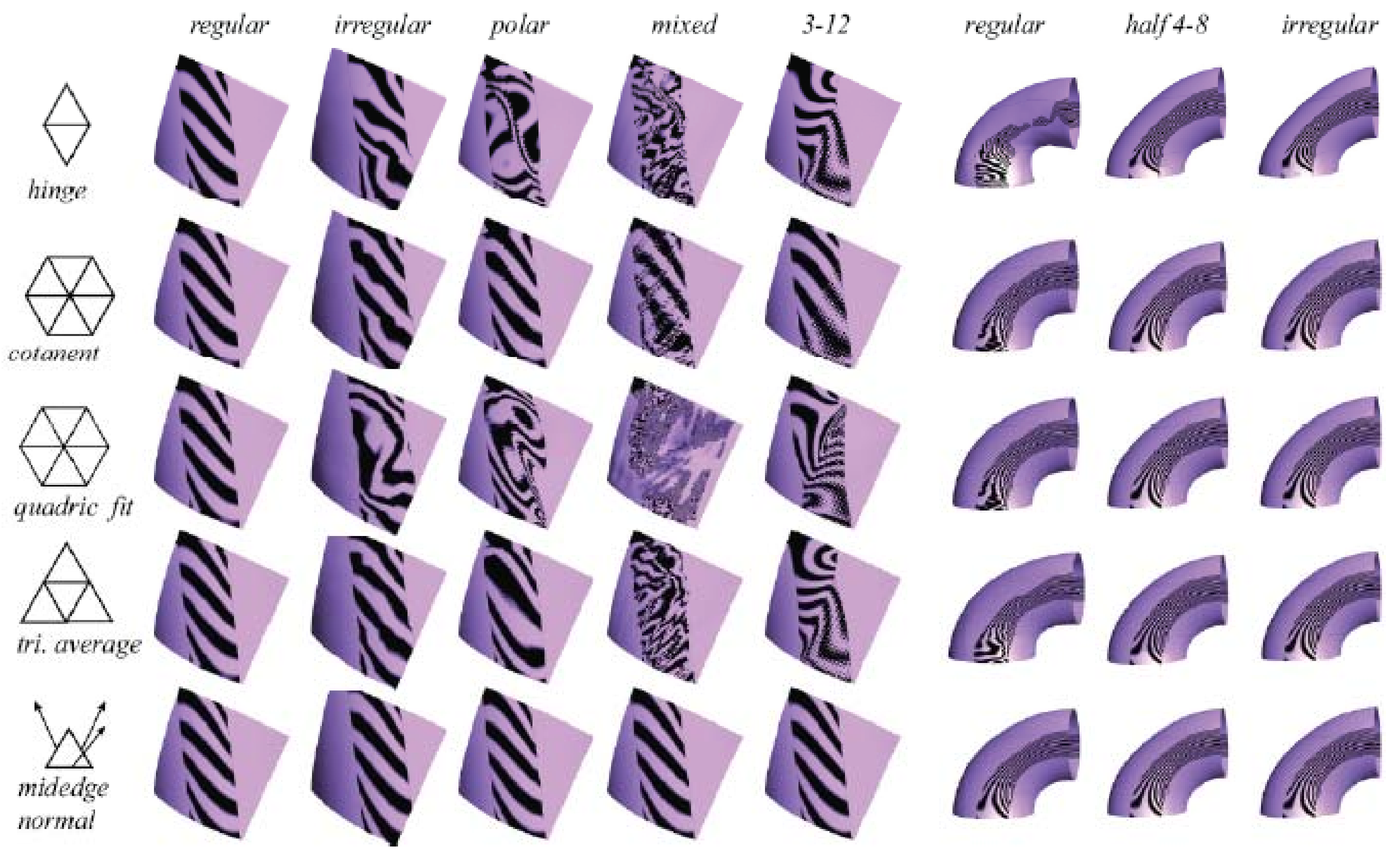
- P. Alliez, *Estimating Curvature Tensors on Triangle Meshes*, Source Code
  - http://www-sop.inria.fr/geometrica/team/Pierre.Alliez/demos/curvature/



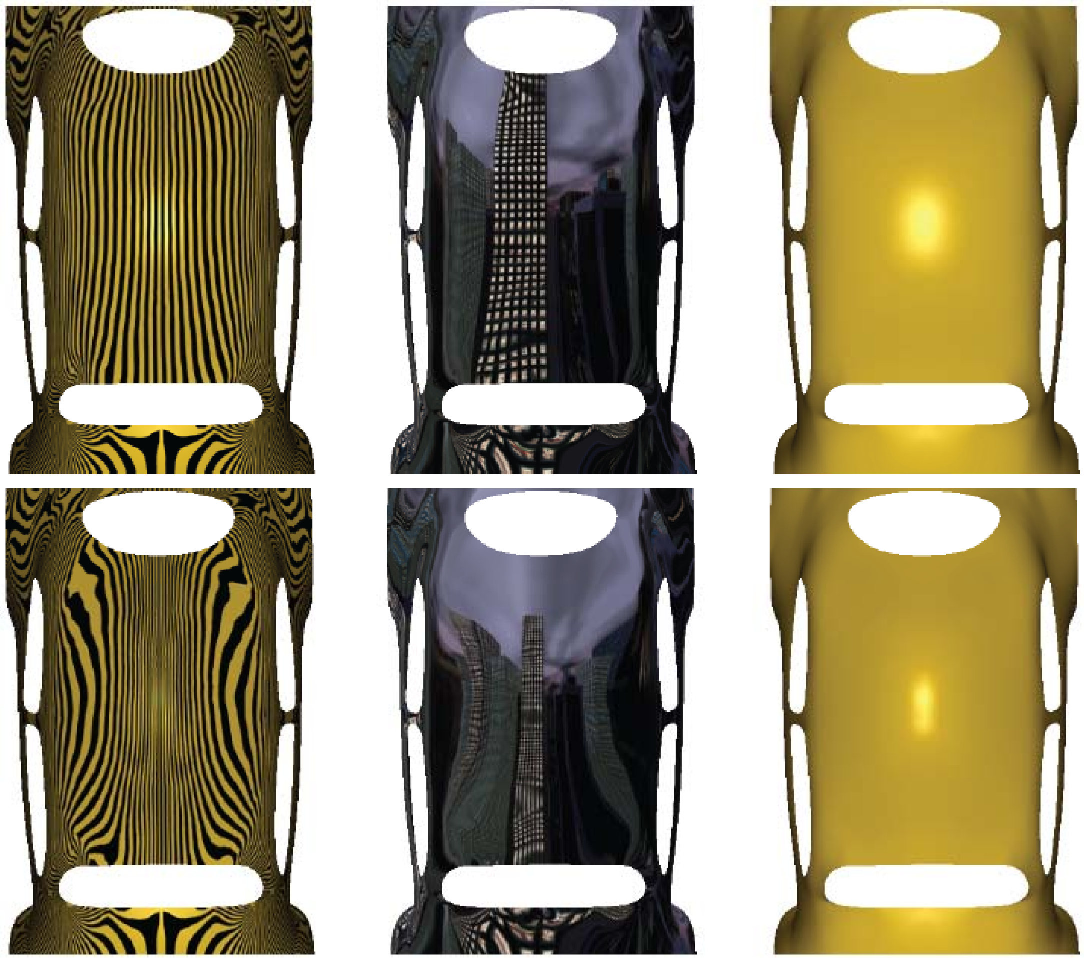
principal directions

Links and literature

- Grinspun et al.: *Computing discrete shape operators on general meshes, Eurographics 2006*

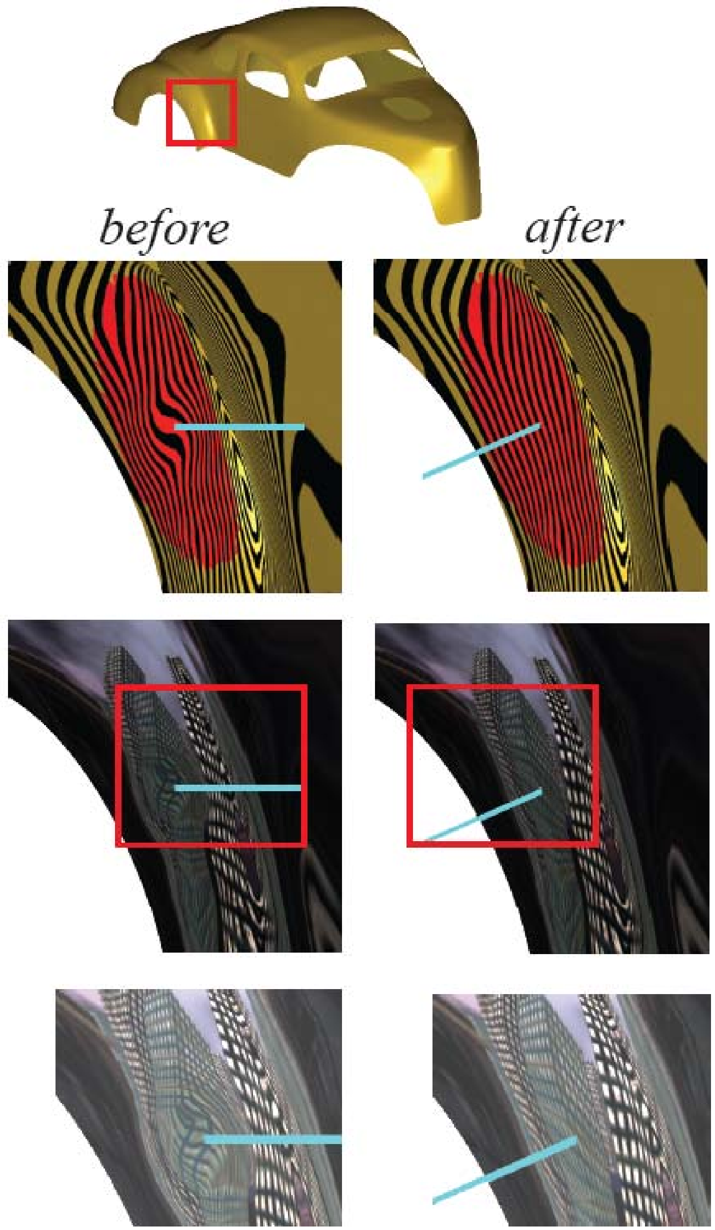


Reflection lines as an inspection tool



[Shape optimization using reflection lines]
E. Tosun, Y. I. Gingold, J. Reisman, D. Zorin
Symposium on Geometry Processing 2007

Reflection lines as an inspection tool



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