3D Geometry for Computer Graphics - Exercise 5

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1. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that A maps the unit sphere $S = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\| = 1\}$ to an ellipsoid. *Guidance*: ellipsoids can be represented as quadratic forms in the following way:

$$E = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^t M \mathbf{x} = 1\},$$

   where $M \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix (i.e. all the eigenvalues of M are positive). For the special case where all the eigenvalues of M are equal, we get a sphere. Take any vector $\mathbf{v}$ that fits the quadratic form of the unit sphere. You want to know where the vectors of the form $\mathbf{w} = A\mathbf{v}$ live. Substitute $\mathbf{v}$ by $A^{-1}\mathbf{w}$ in the quadratic form of the unit sphere, and you will arrive at the quadratic form on which $\mathbf{w}$ lives. Prove that its matrix is positive definite and you are done.

2. In geometry and in Computer Graphics, we describe our objects by representing them in some coordinate systems. When we describe 3D scenes, we use some basis of $\mathbb{R}^3$ and represent the objects with respect to this basis. Usually, **orthonormal** bases are used. Think of reasons why. Why do people prefer to use unit-length perpendicular vectors as the coordinate axes and not some vectors of general (non-equal) lengths and with general different angles between them?