3D Geometry for Computer Graphics

Exercise 5 – selected solutions

1. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that A maps the unit sphere $S = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\| = 1\}$ to an ellipsoid. *Guidance*: ellipsoids can be represented as quadratic forms in the following way:

$$E = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}^t M \mathbf{x} = 1\},$$

where $M \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix (i.e. all the eigenvalues of M are positive). For the special case where all the eigenvalues of M are equal, we get a sphere. Take any vector $\mathbf{v}$ that fits the quadratic form of the unit sphere. You want to know where the vectors of the form $\mathbf{w} = A\mathbf{v}$ live. Substitute $\mathbf{v}$ by $A^{-1}\mathbf{w}$ in the quadratic form of the unit sphere, and you will arrive at the quadratic form on which $\mathbf{w}$ lives. Prove that its matrix is positive definite and you are done.

Answer: We know $\mathbf{v}$ lives on the unit sphere, so

$$\|\mathbf{v}\| = 1 \quad \Rightarrow \quad \|\mathbf{v}\|^2 = 1 \quad \Rightarrow \quad \mathbf{v}^T\mathbf{v} = 1$$

Substitute $\mathbf{v}$ by $A^{-1}\mathbf{w}$:

$$\begin{aligned} \mathbf{v}^T\mathbf{v} &= 1 \\ (A^{-1}\mathbf{w})^T(A^{-1}\mathbf{w}) &= 1 \\ \mathbf{w}^T\left((A^{-1})^T(A^{-1})\right)\mathbf{w} &= 1 \end{aligned}$$

To prove that all $\mathbf{w}$ live on an ellipsoid, we need to show that the matrix $M = (A^{-1})^T(A^{-1})$ is positive definite. Denote $S = A^{-1}$, so that $M = S^T S$. Matrix M is positive definite if $\forall\mathbf{u} \neq 0\ \langle M\mathbf{u}, \mathbf{u}\rangle > 0$.

$$\langle M\mathbf{u}, \mathbf{u}\rangle = (M\mathbf{u})^T\mathbf{u} = \mathbf{u}^T M\mathbf{u} = \mathbf{u}^T S^T S\mathbf{u} = (S\mathbf{u})^T(S\mathbf{u}) = \|S\mathbf{u}\|^2 > 0.$$

Therefore M is positive definite and thus all $\mathbf{w}$ live on an ellipsoid. So, our linear transformation A maps the unit sphere to an ellipsoid.