SYMPOSIUM ON GEOMETRY PROCESSING 2020

UTRECHT NL

Properties of Laplace Operators for Tetrahedral Meshes

Philipp Herholz, Olga Sorkine Hornung   ETH Zurich

Marc Alexa, Maximilian Kohlbrenner   TU Berlin

Discrete Laplacian on Simplicial Meshes

- Many applications in geometry processing:
  deformation, parameterization, heat diffusion, uniformization.

- All these examples have direct analogies for tetrahedral meshes.

Discrete Laplacian on Simplicial Meshes

- All Laplacians we consider here are given by weights w_{ij} .

$$(\mathbf{Lf})_i = \sum_{(i,j)\in\mathcal{M}} w_{ij}(f_j - f_i)$$

$$w_{ij} = w_{ji}$$

- Properties by construction:
  (weak) locality, symmetry, constants are part of the null space.

Discrete Laplacian on Simplicial Meshes

- Additional desirable properties:

negative semi-definite

Dirichlet energy $-\mathbf{f}^\top \mathbf{L}\mathbf{f}$ is non-negative for any function $\mathbf{f}$.

positive coefficients w_{ij}

Maximum principle.

linear precision

$(\mathbf{L}\mathbf{f})_i = 0$ for inner vertices i and linear functions $\mathbf{f}$.

Discrete Laplacian on Simplicial Meshes

- Discrete Laplacians are widely studied for triangle meshes.
- Arguably the most common choice:

cotan Laplacian

(negative semi-definite, symmetric, linear precision, local)

- If maximum principle is needed for non-Delaunay meshes:

intrinsic cotan Laplacian

(negative semi-definite, symmetric, linear precision, positivity)

Discrete Laplacian on Simplicial Meshes

- Is there a similar default option for simplicial meshes?

Short answer: No

<u>Triangle meshes</u>

FEM → Cotan-Laplacian

FV/DEC → Cotan-Laplacian

<u>Simplicial meshes</u>

FEM → Primal Laplacian

FV/DEC → Dual Laplacian

Primal Laplacian $\neq$ Dual Laplacian

Discrete Laplacian on Simplicial Meshes

- Is there a similar default option for simplicial meshes?

Detailed answer:

Eurographics Symposium on Geometry Processing 2020
Q. Huang and A. Jacobson
(Guest Editors)

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Properties of Laplace Operators for Tetrahedral Meshes

Marc Alexa[1] Philipp Herholz[2] Maximilian Kohlbrenner[1] Olga Sorkine-Hornung[2]

[1] TU Berlin, Germany [2] ETH Zurich, Switzerland

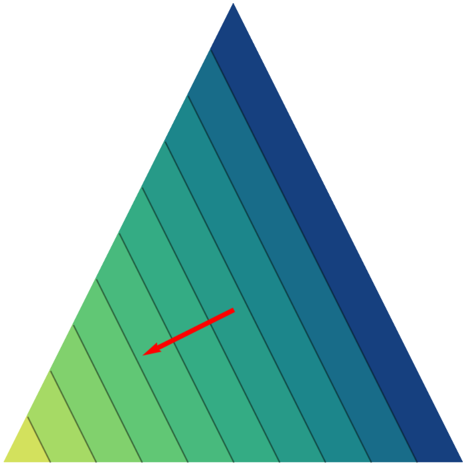
Abstract

Discrete Laplacians for triangle meshes are a fundamental tool in geometry processing. The so-called cotan Laplacian is widely used since it preserves several important properties of its smooth counterpart. It can be derived from different principles: either considering the piecewise linear nature of the primal elements or associating values to the dual vertices. Both approaches lead to the same operator in the two-dimensional setting. In contrast, for tetrahedral meshes, only the primal construction is reminiscent of the cotan weights, involving dihedral angles. We provide explicit formulas for the lesser-known dual construction. In both cases, the weights can be computed by adding the contributions of individual tetrahedra to an edge. The resulting two different discrete Laplacians for tetrahedral meshes only retain some of the properties of their two-dimensional counterpart. In particular, while both constructions have linear precision, only the primal construction is positive semi-definite and only the

Discrete Laplacian: Linear Finite Elements

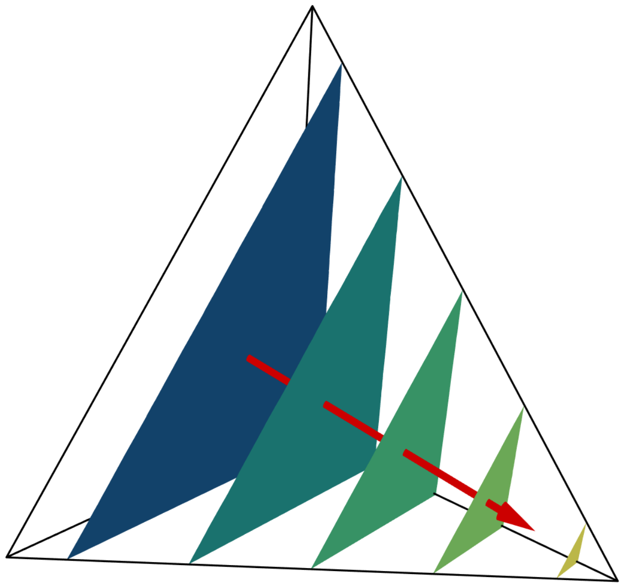
Discrete Laplacian: Linear Finite Elements

- Each vertex v_i has a unique function b_i that is
  - linear on the elements (triangles/tetrahedra).
  - fulfills $b_i(v_j) = \delta_{ij}$.



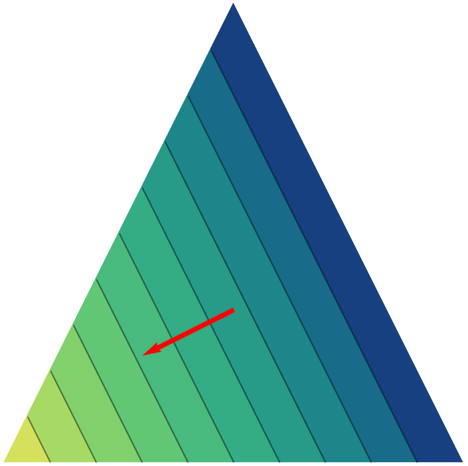
- Represent function in this basis

$$f = \sum_i f_i b_i$$



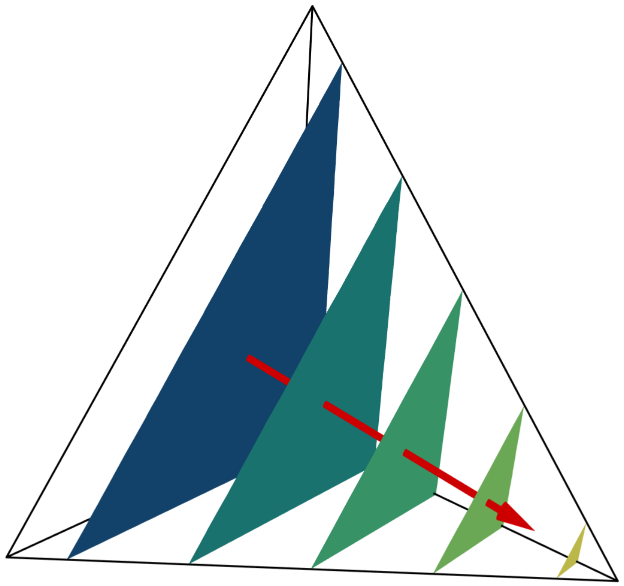
Discrete Laplacian: Linear Finite Elements

- Gradients of b_i are constant on elements.



- Gradient of piecewise linear function by linearity.

$$\nabla f = \nabla \sum_i f_i b_i = \sum_i f_i \nabla b_i$$



- A matrix $\mathbf{G}$ maps vertex values to gradients.

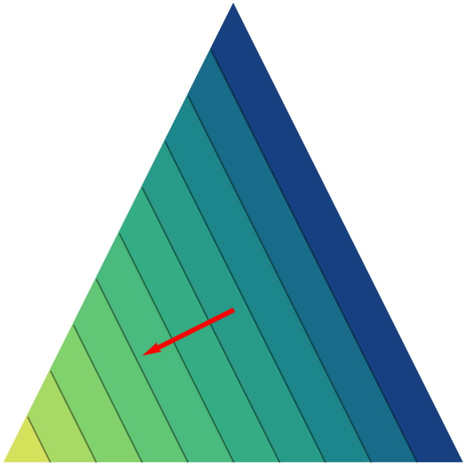
Discrete Laplacian: Linear Finite Elements

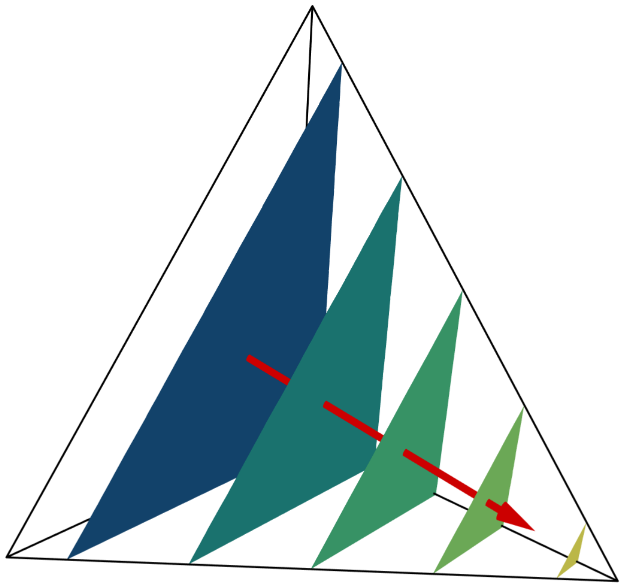
- Divergence is the adjoint of the gradient

$$\mathbf{D} = -\mathbf{G}^\top \mathbf{A}$$

- $\mathbf{A}$ is a diagonal matrix containing areas/volumes.
- The Laplacian can be defined as divergence of the gradient:

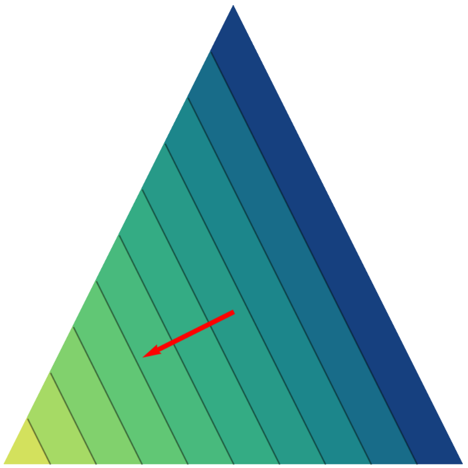
$$\mathbf{L} = -\mathbf{G}^\top \mathbf{A} \mathbf{G}$$





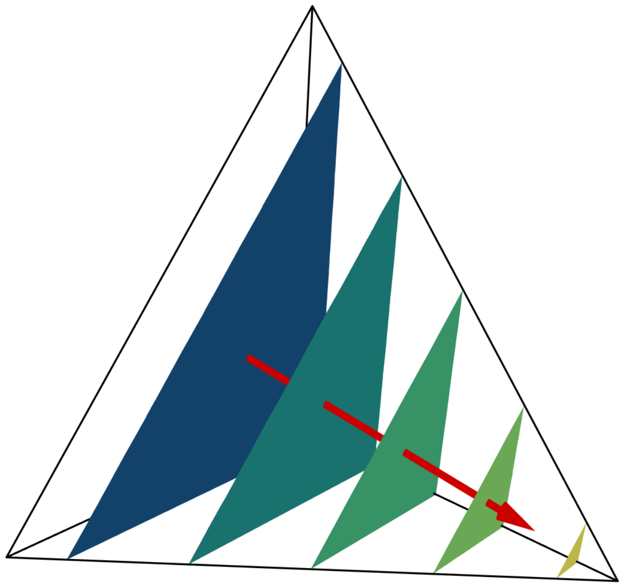
Discrete Laplacian: Linear Finite Elements

$$\mathbf{L} = -\mathbf{G}^\top \mathbf{A} \mathbf{G}$$



- Negative semi-definite by construction.
- Linear functions are exactly reproduced.

$\Rightarrow$ linear precision: the Laplacian of linear functions vanishes (at interior vertices).

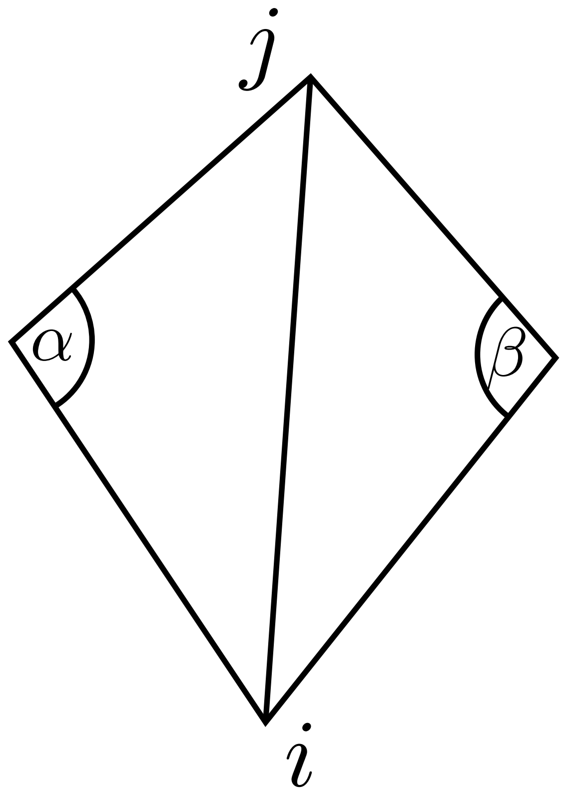


Discrete Laplacian: Linear Finite Elements

$$\mathbf{L} = -\mathbf{G}^\top \mathbf{A} \mathbf{G}$$

- Yields cotan Laplacian for triangle meshes.

$$w_{ij} = \frac{1}{2} \cot \alpha + \frac{1}{2} \cot \beta$$

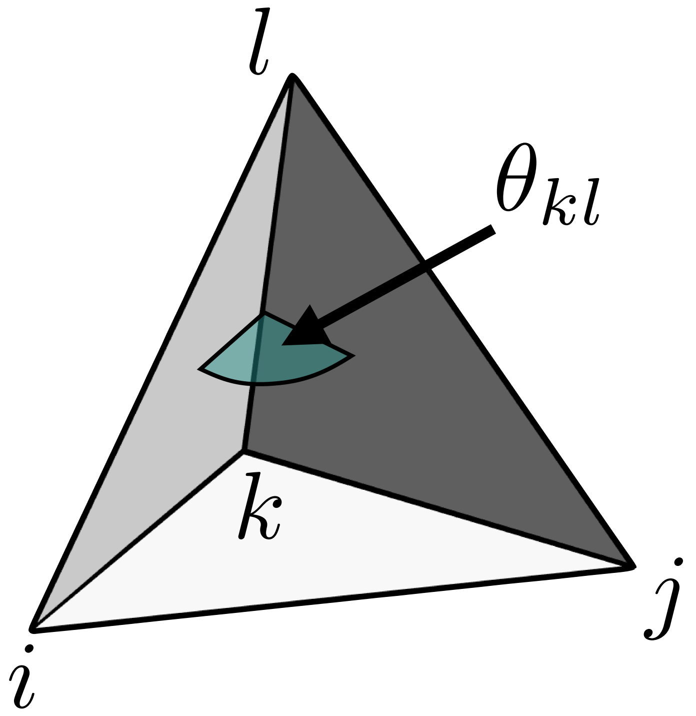


- Construction per triangle.

Discrete Laplacian: Linear Finite Elements

- Analogous formula for tetrahedral meshes

$$w_{ij} = \frac{1}{6} \sum_{(ijkl) \in \mathcal{M}} \|\mathbf{v}_k - \mathbf{v}_l\| \cot \theta_{kl}$$



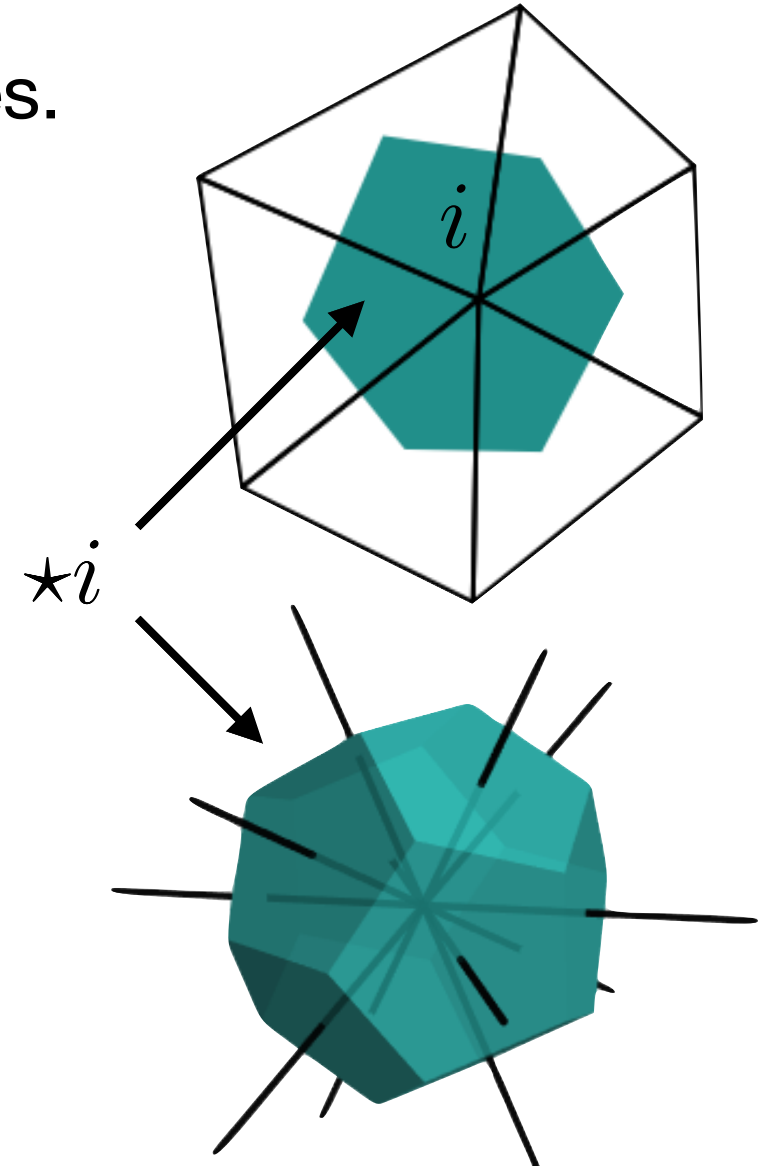
- Construction by summing up per element: *strong* locality.

Discrete Laplacian: Finite Volumes

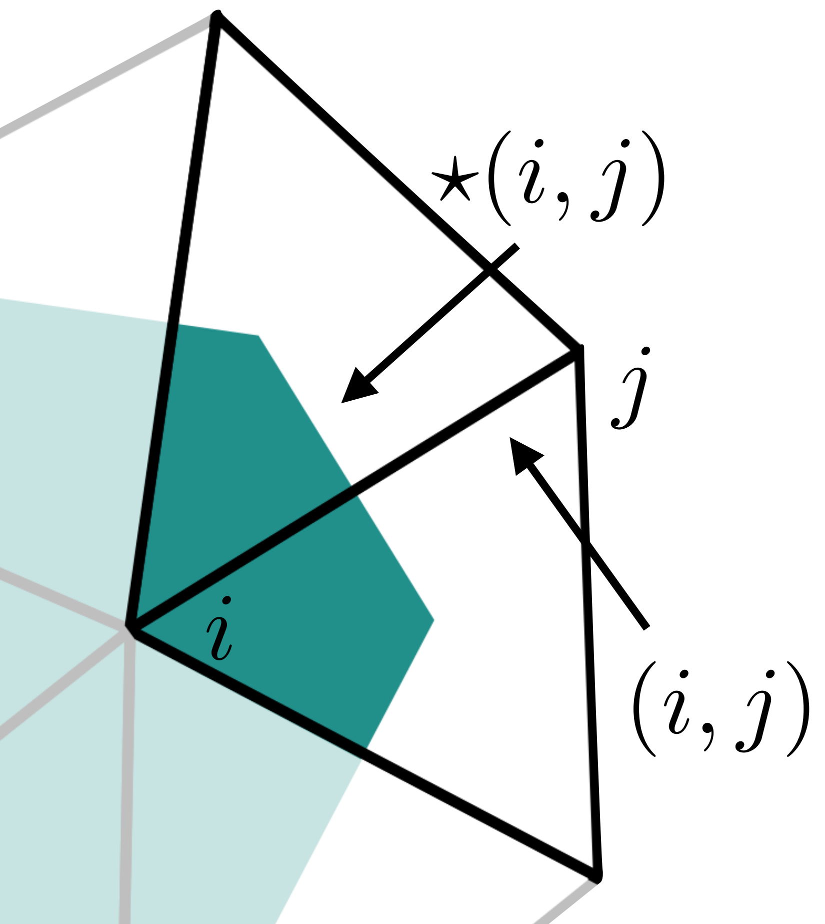
Discrete Laplacian: Finite Volumes

- Associate constant values to volumes at vertices.
- Use circumcentric dual volumes $\star i$.

$$\int_{\star i} \Delta f \; dV = \int_{\star i} \nabla(\nabla f) \; dV = \int_{\partial \star i} (\nabla f) \cdot \mathbf{n} \; dS$$



Discrete Laplacian: Finite Volumes

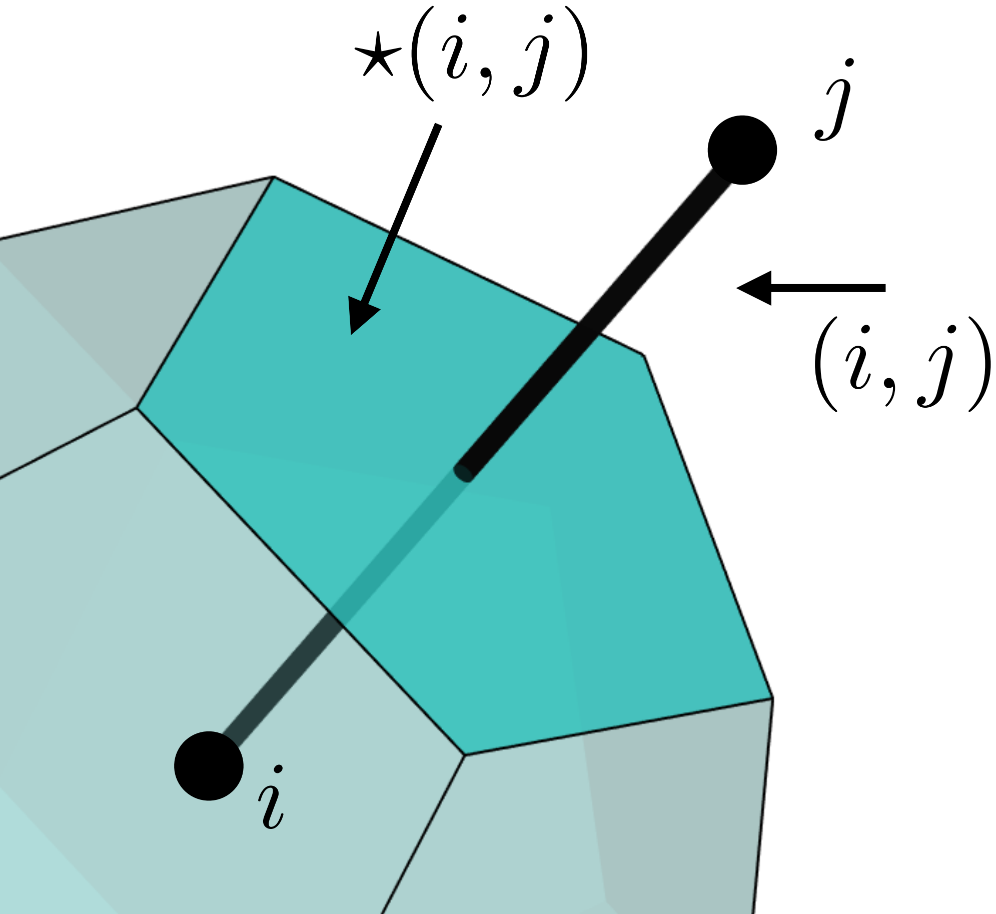


$$\int_{\star i} \Delta f \; dV = \int_{\partial \star i} (\nabla f) \cdot \mathbf{n} \; dS$$

$$\sum_{(ij)\in\mathcal{M}} \frac{(f_j - f_i)}{\mathrm{Vol}(i,j)} \mathrm{Vol}(\star(i,j))$$

$$w_{ij} = \frac{\mathrm{Vol}(\star(i,j))}{\mathrm{Vol}(i,j)}$$

Discrete Laplacian: Finite Volumes



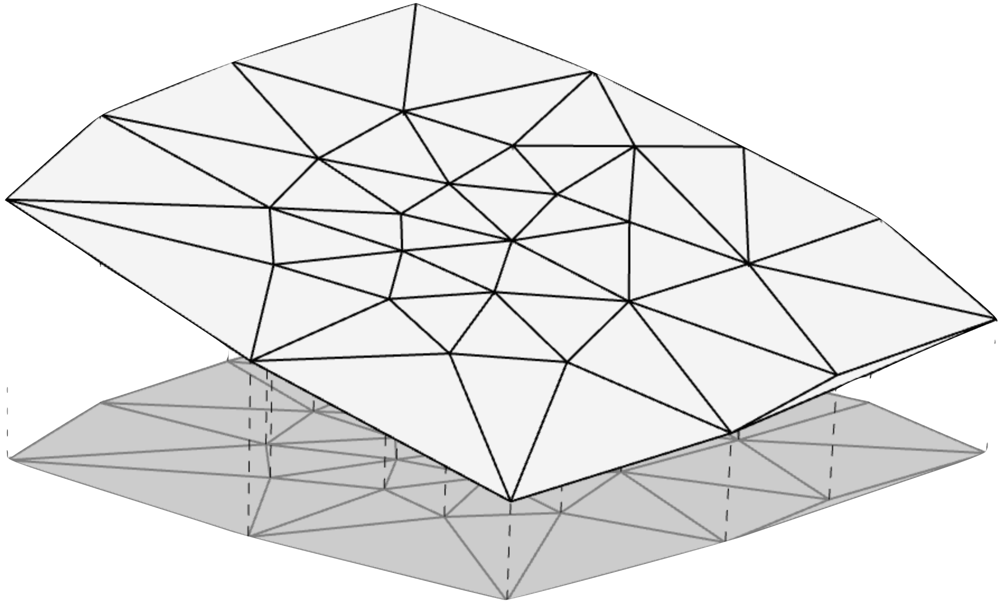
$$\int_{\star i} \Delta f \, dV = \int_{\partial \star i} (\nabla f) \cdot \mathbf{n} \, dS$$

$$\sum_{(ij)\in\mathcal{M}} \frac{(f_j - f_i)}{\mathrm{Vol}(i,j)} \mathrm{Vol}(\star(i,j))$$

$$w_{ij} = \frac{\mathrm{Vol}(\star(i,j))}{\mathrm{Vol}(i,j)}$$

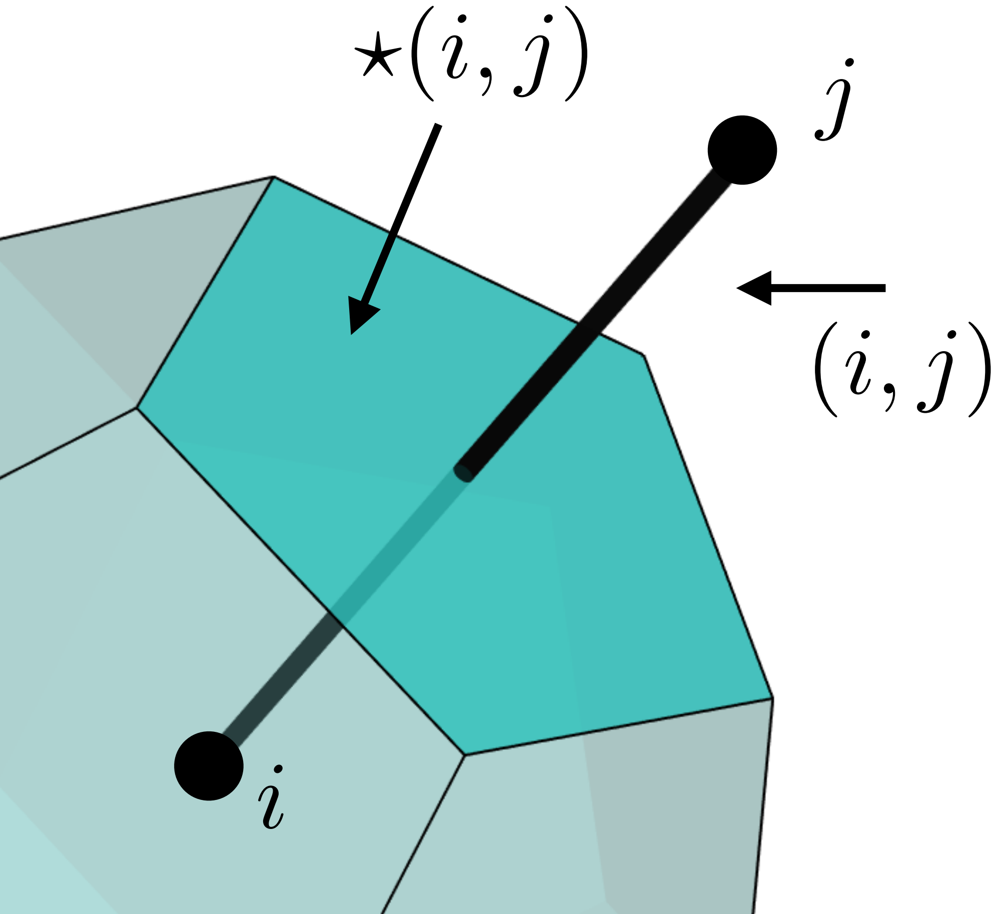
Discrete Laplacian: Finite Volumes

- The set $(\mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ spans the space of linear functions.
- Constants are part of the kernel by default.
- Linear precision: $(\mathbf{L}\mathbf{X})_i = 0$ for the vertex positions $\mathbf{X} \in \mathbb{R}^{n \times 3}$.



Discrete Laplacian: Finite Volumes

- Linear precision: $(\mathbf{LX})_i = 0$ for the vertex positions $\mathbf{X} \in \mathbb{R}^{n\times 3}$.

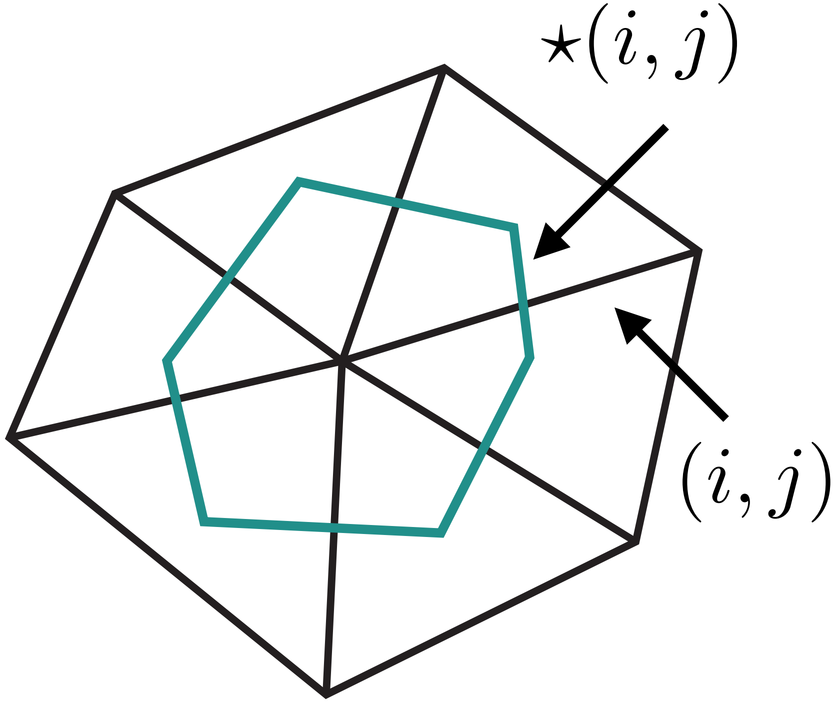


$$\sum_{(ij)\in\mathcal{M}} \frac{(\mathbf{v}_j - \mathbf{v}_i)}{\|\mathbf{v}_j - \mathbf{v}_i\|} \mathrm{Vol}(\star(i,j)) = 0$$

Vector area integrated over a closed surface vanishes.

Discrete Laplacian: Finite Volumes

- Positivity:

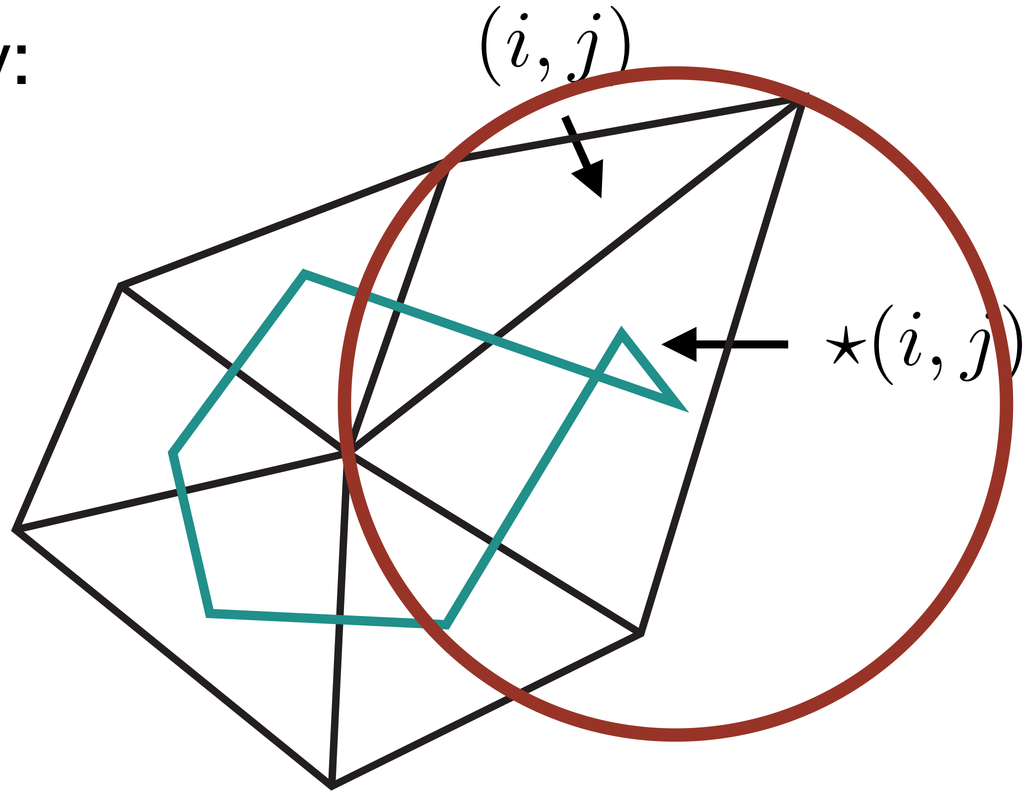


$$w_{ij} = \frac{\mathrm{Vol}(\star(i,j))}{\mathrm{Vol}(i,j)}$$

- The vector $\dfrac{\mathbf{v}_j - \mathbf{v}_i}{\|\mathbf{v}_j - \mathbf{v}_i\|}$ is the *outward* directed boundary normal.

Discrete Laplacian: Finite Volumes

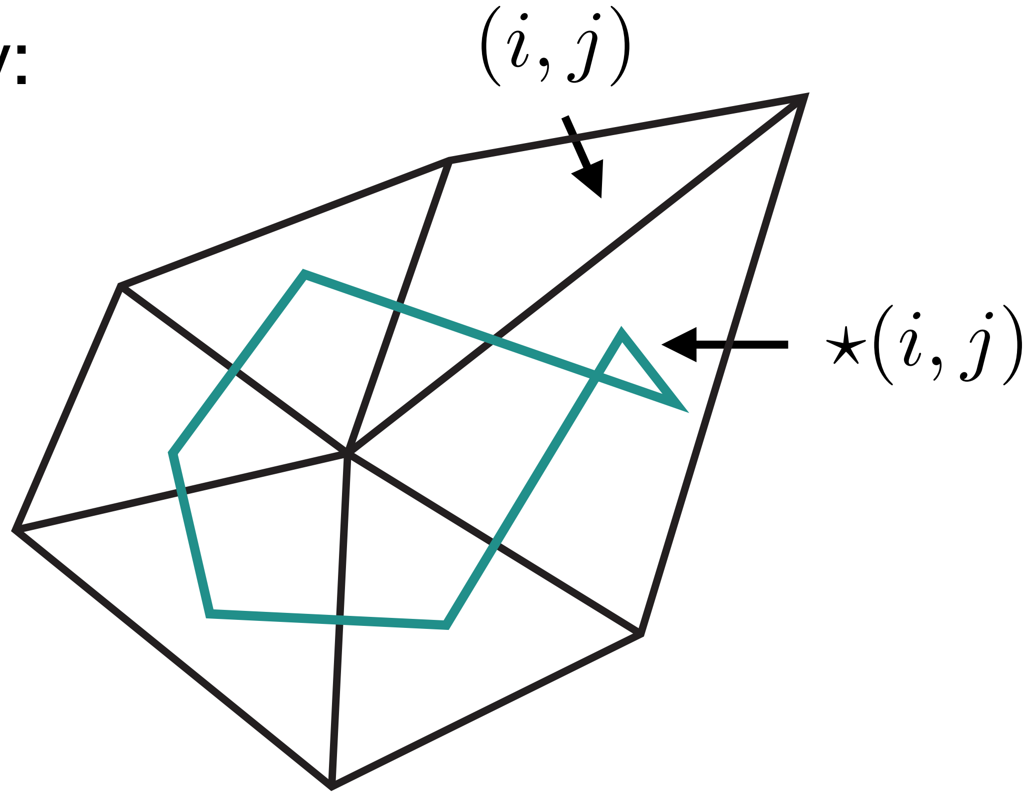
- Positivity:



$$w_{ij} = \frac{\mathrm{Vol}(\star(i,j))}{\mathrm{Vol}(i,j)}$$

Discrete Laplacian: Finite Volumes

- Positivity:

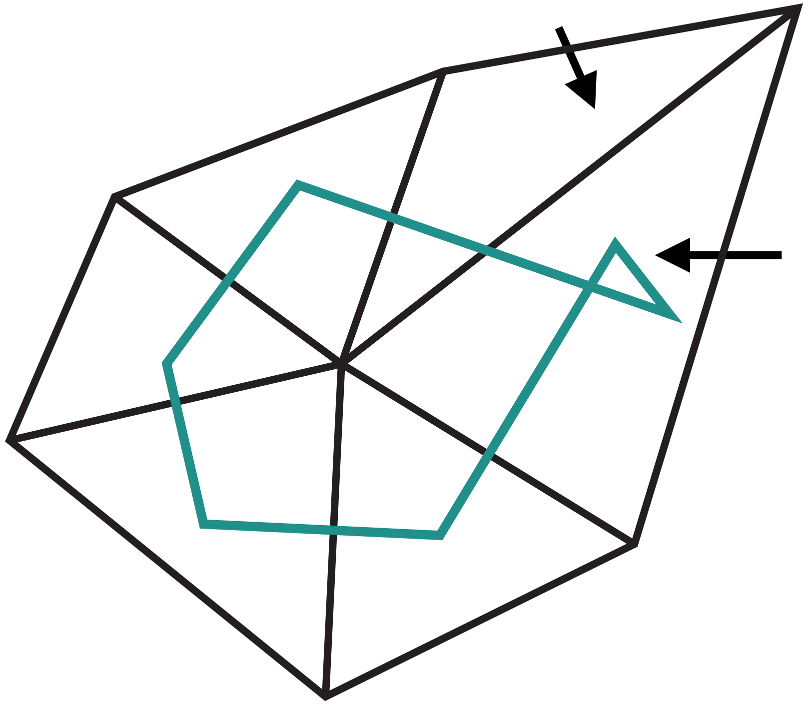


$$w_{ij} = \frac{\mathrm{Vol}(\star(i,j))}{\mathrm{Vol}(i,j)}$$

- The vector $\dfrac{\mathbf{v}_j - \mathbf{v}_i}{\|\mathbf{v}_j - \mathbf{v}_i\|}$ is the *inward* directed boundary normal.

Discrete Laplacian: Finite Volumes

- Positivity:



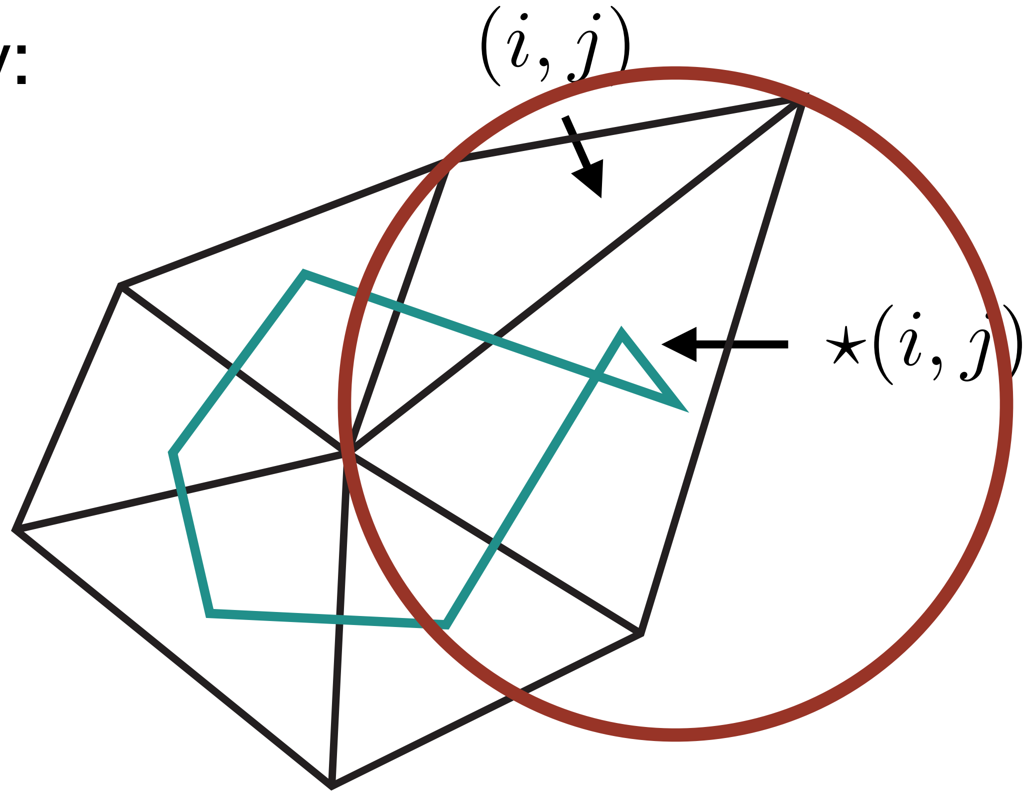
$$w_{ij} = \frac{\mathrm{Vol}(\star(i,j))}{\mathrm{Vol}(i,j)}$$

negative dual volume

- The vector $\frac{\mathbf{v}_j - \mathbf{v}_i}{\|\mathbf{v}_j - \mathbf{v}_i\|}$ is the *inward* directed boundary normal.

Discrete Laplacian: Finite Volumes

- Positivity:



$$w_{ij} = \frac{\text{Vol}(\star(i,j))}{\text{Vol}(i,j)}$$

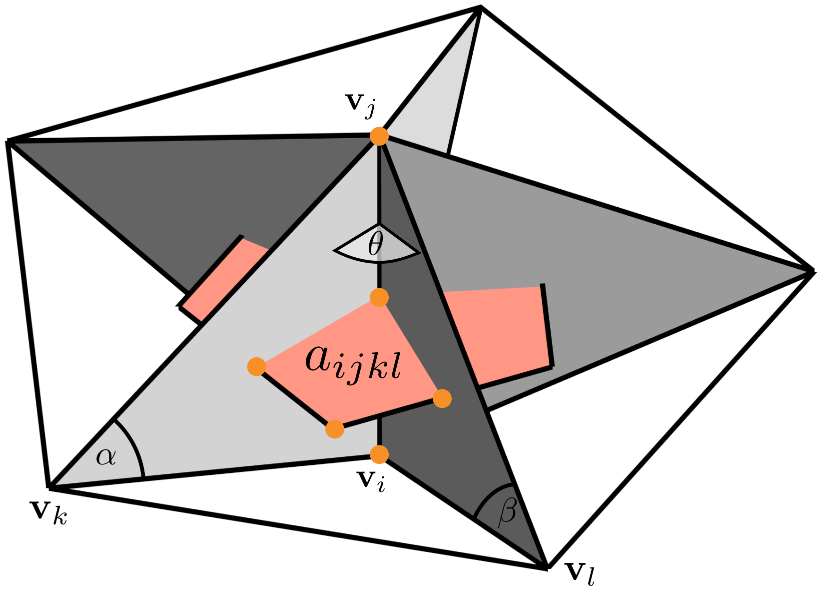
- Delaunay meshes have embedded circumcentric duals => $w_{ij} \geq 0$

Discrete Laplacian: Strong Locality

- Element wise construction:

$$w_{ij} = \frac{\|\mathbf{v}_i - \mathbf{v}_j\|}{8} \sum_{ijkl} \cot\theta \left(2\frac{\cot\alpha \ \cot\beta}{\cos\theta} - \cot^2\alpha - \cot^2\beta \right)$$

- Applies to non-Delaunay meshes.
- Easy boundary treatment.



Discrete Laplacian: Properties

- Properties by construction (finite elements/finite volumes):

symmetry, strong locality, linear precision

- Properties by construction (finite elements):

negative semi-definite

- Properties by construction (finite volumes, Delaunay mesh):

positive coefficients (mesh interior)

Discrete Laplacian: Properties 2d

- For triangle meshes:

finite element Laplace = finite volume Laplace = cotan Laplace

- Delaunay meshes:

negative semi-definite and positive coefficients (interior)

- non-Delaunay meshes:

negative semi-definite

Discrete Laplacian: Properties 3d

- For tetrahedral meshes:

finite element Laplace $\neq$ finite volume Laplace

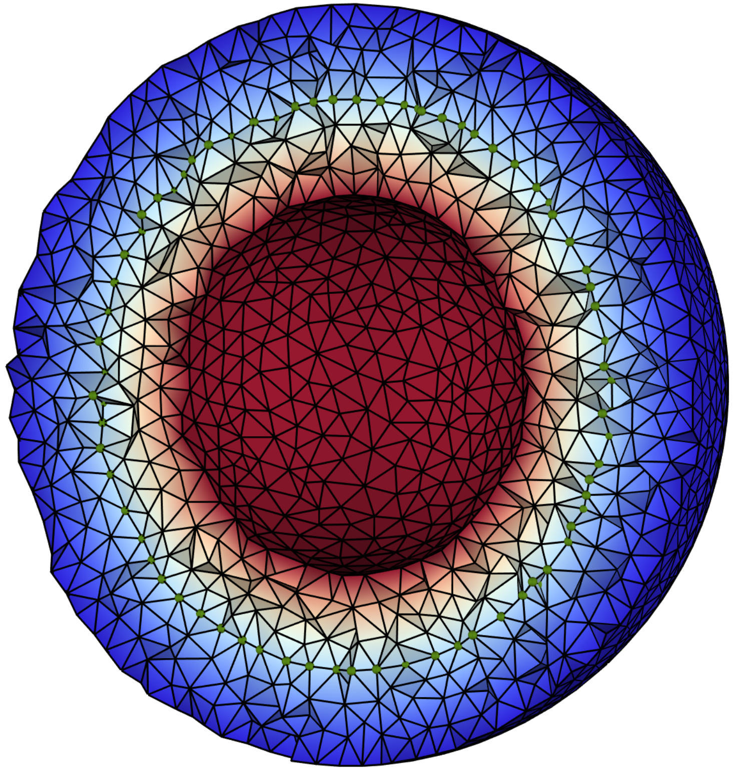
Discrete Laplacian: Properties 3d

- Finite volume Laplace:

Delaunay mesh:	positive weights (away from the boundary)
non-Delaunay mesh:	-

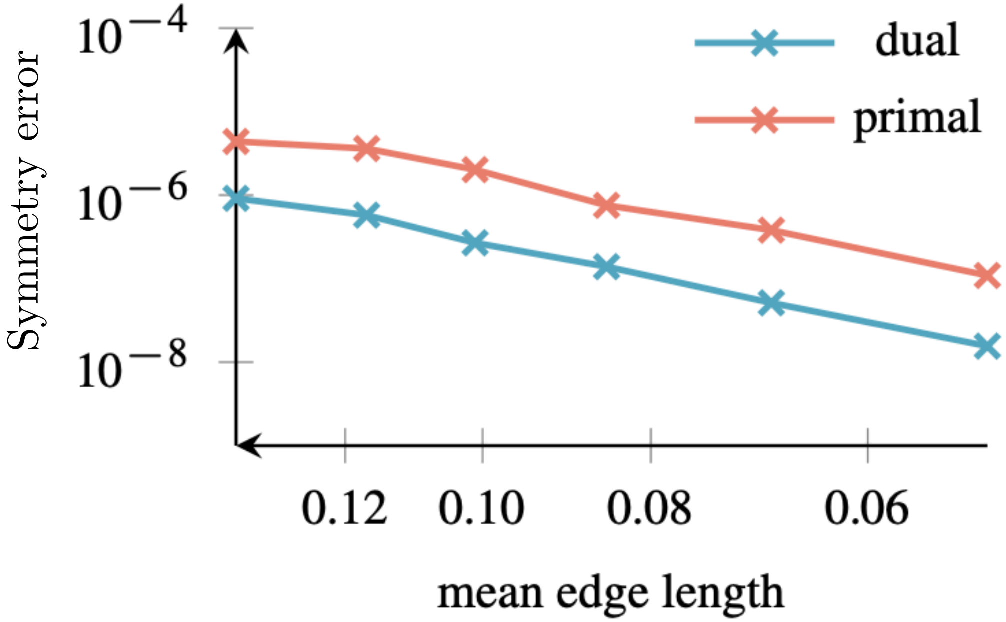
Numerical Experiments

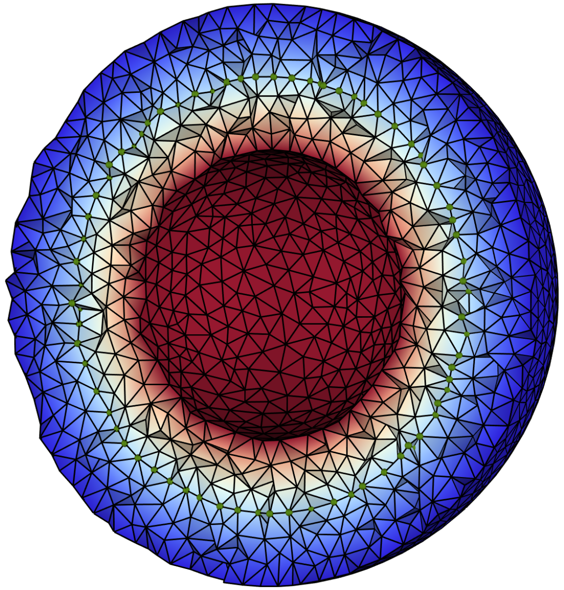
- Minimizing Dirichlet energy: Delaunay mesh



Numerical Experiments

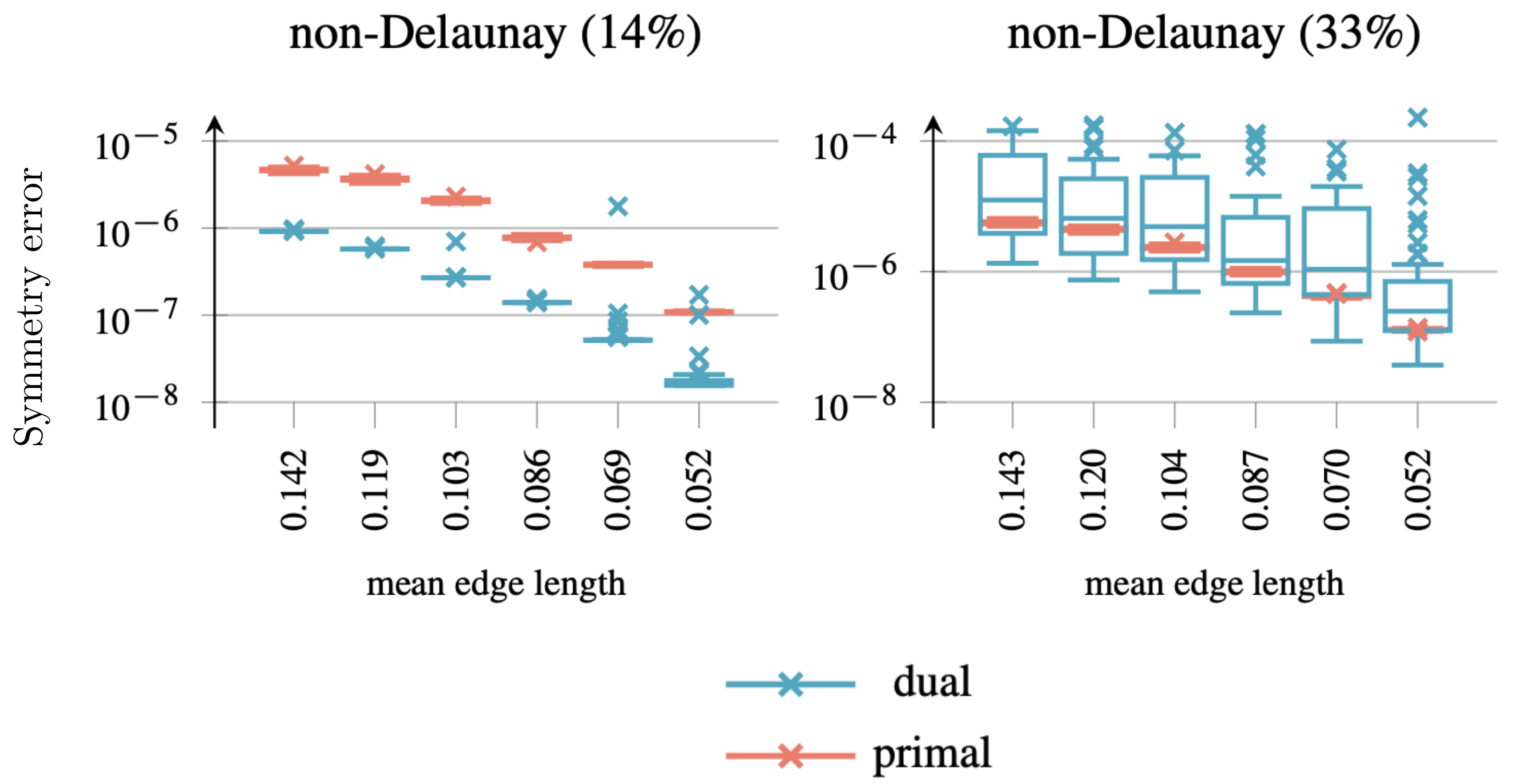
- Minimizing Dirichlet energy: Delaunay mesh

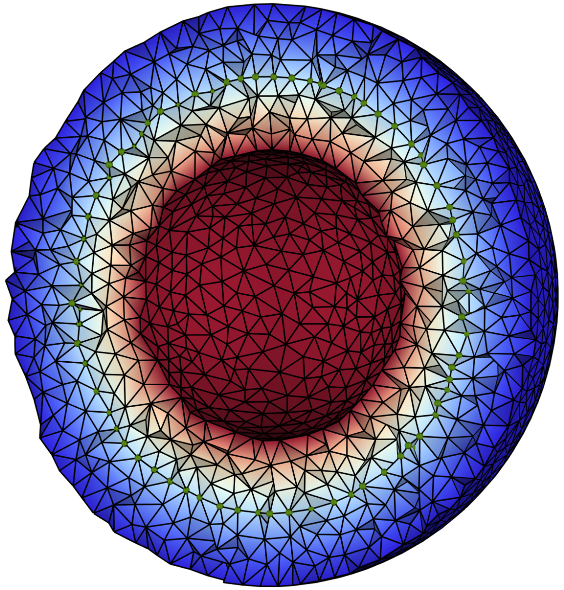




Numerical Experiments

- Minimizing Dirichlet energy: non-Delaunay mesh





Conclusion

- The choice of Laplacian for simplicial meshes depends on:
  - mesh type
  - problem type
  - numerical method

Thank you.