

A Cotangent Laplacian for Images as Surfaces

Alec Jacobson Olga Sorkine
ETH Zurich

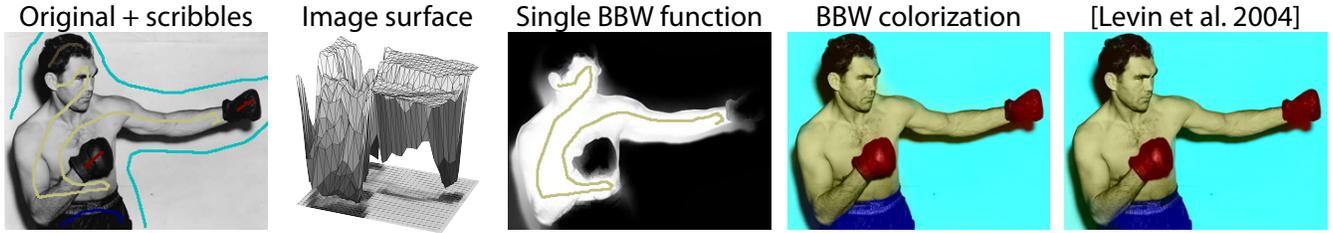


Figure 1: Left to right: an image with color scribbles is fit with a mesh and lifted into \mathbb{R}^3 according to intensity values. Bounded Biharmonic Weights computed for each scribble (shown for body) are used to colorize the image. Compare to colorization by [Levin et al. 2004].

Abstract

By embedding images as surfaces in a high dimensional coordinate space defined by each pixel's Cartesian coordinates and color values, we directly define and employ cotangent-based, discrete differential-geometry operators. These operators define discrete energies useful for image segmentation and colorization.

Keywords: image processing, discrete differential geometry

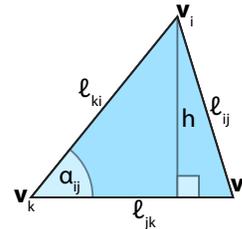
Many image processing techniques rely on differential operators defined in terms of some metric adapted to image content. For example, discrete Laplacians with stencils weighted by a function of pixel locations and color values define energies whose minima intuitively propagate tone map adjustments [Lischinski et al. 2006] or sparse color values [Levin et al. 2004; Lischinski et al. 2006] imply discrete Laplacians, they are rarely labeled as such. As a result they appear to be less studied in this regard and most likely only enjoy a subset of these properties. However, they depend on the image color values, and are thus tantamount to defining a Laplacian in terms of some content-adaptive metric. As of yet, the planar triangle meshes previously used in image processing incorporate only the Cartesian coordinates of mesh vertices and are thus defined solely by the Euclidean image-plane metric, ignorant of image content.

The mesh-based Laplacians enjoy well-studied properties: convergence w.r.t. mesh resolution, positive semi-definiteness when defining Dirichlet energies, and symmetric, locally-supported stencil weights [Wardetzky et al. 2007]. Though the content-adaptive stencils used by [Levin et al. 2004; Lischinski et al. 2006] imply discrete Laplacians, they are rarely labeled as such. As a result they appear to be less studied in this regard and most likely only enjoy a subset of these properties. However, they depend on the image color values, and are thus tantamount to defining a Laplacian in terms of some content-adaptive metric. As of yet, the planar triangle meshes previously used in image processing incorporate only the Cartesian coordinates of mesh vertices and are thus defined solely by the Euclidean image-plane metric, ignorant of image content.

We define an *image surface* by first overlaying a triangle mesh atop the image. We place vertices at pixel centers and Delaunay-triangulate them. The mesh is then *lifted* into higher dimensional space by appending each pixel's color values as coordinates to the corresponding mesh vertex. Thus the pixel i at location (x_i, y_i) is lifted to (x_i, y_i, I_i) , $(x_i, y_i, R_i, G_i, B_i)$, or $(x_i, y_i, L_i, A_i, B_i)$ in a grayscale, RGB, or LAB color model respectively.

Now our triangle mesh lives as a disk-topology surface embedded in a higher dimension. If only a single color channel is used then the surface lives in \mathbb{R}^3 , and typical discrete differential operators common in 3D mesh processing may be immediately applied. If

we use more channels then our surface lives in \mathbb{R}^5 or possibly higher. At first glance it may seem difficult to define the usual operators in higher dimensions. However, the building blocks of standard operators, e.g. the discrete Laplace-Beltrami operator, are the triangle areas and cotangents of each triangle corner angle [Meyer et al. 2003], and these may be defined *intrinsically* based solely on triangle edge lengths (rather than using cross products as one might in \mathbb{R}^3).



Consider a triangle with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k \in \mathbb{R}^d$. The triangle area A_{ijk} is defined intrinsically by [Heron 60]:

$$A_{ijk} = \sqrt{r(r - l_{ij})(r - l_{jk})(r - l_{ki})}$$

where l_{ij} is the length of the edge between \mathbf{v}_i and \mathbf{v}_j , and r is the semi-perimeter $\frac{1}{2}(l_{ij} + l_{jk} + l_{ki})$.

We may similarly define the cotangent of the angle opposite each edge. First we can derive the cosine and sine. Recall the law of cosines:

$$l_{ij}^2 = l_{jk}^2 + l_{ki}^2 - 2l_{jk}l_{ki} \cos \alpha_{ij} \rightarrow \cos \alpha_{ij} = \frac{-l_{ij}^2 + l_{jk}^2 + l_{ki}^2}{2l_{jk}l_{ki}}$$

For sine, we employ the familiar area formula treating the $\overline{\mathbf{v}_j \mathbf{v}_k}$ as base:

$$A_{ijk} = \frac{1}{2}l_{jk}l_{ki} \sin \alpha_{ij} \rightarrow \sin \alpha_{ij} = \frac{2A_{ijk}}{l_{jk}l_{ki}}$$

Finally putting these together we have:

$$\cot \alpha_{ij} = \frac{\cos \alpha_{ij}}{\sin \alpha_{ij}} = \frac{-l_{ij}^2 + l_{jk}^2 + l_{ki}^2}{2l_{jk}l_{ki}} \frac{l_{jk}l_{ki}}{2A_{ijk}} = \frac{-l_{ij}^2 + l_{jk}^2 + l_{ki}^2}{4A_{ijk}}$$

Note that a similar intrinsic derivation is given in Equations 7 and 13 of [Meyer et al. 2003].

With cotangents in hand, we may employ operators like the discrete Laplacian to minimize the Dirichlet energy over an image surface. Consider the colorization problem. We wish to propagate the colors of sparse user scribbles to the rest of the image in a localized and smooth manner that takes into account the image content. [Levin et al. 2004] pose this as a discrete energy minimization problem. Despite their published formulas, discussion with one of the authors and their published code agree that their energy is a Dirichlet energy resulting in a second-order PDE with a discrete Laplacian as the system matrix, similar to that of [Lischinski et al. 2006].

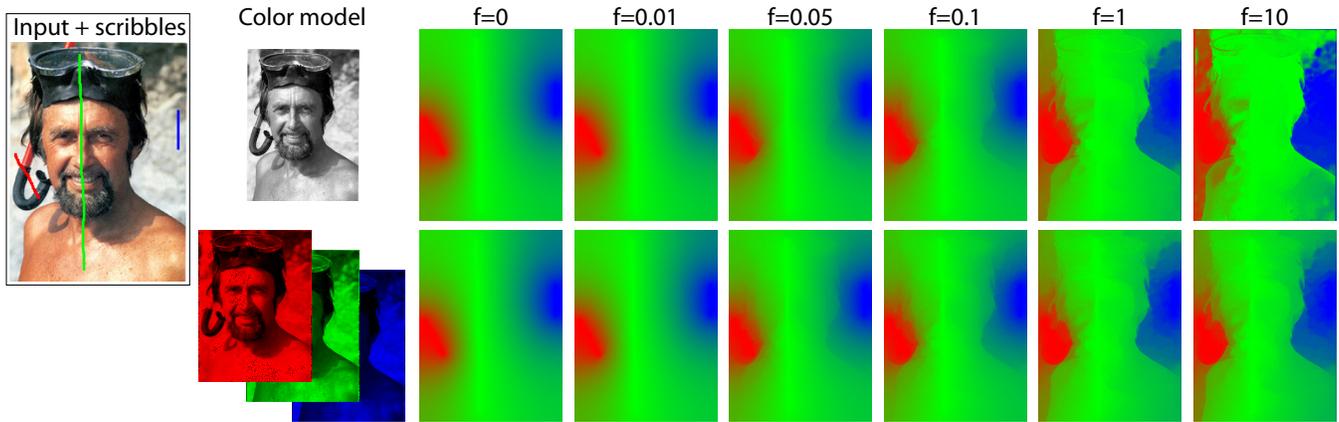


Figure 2: The original input image with colored user scribbles (left inset) defines an image surface in \mathbb{R}^3 using intensity values (top row), or in \mathbb{R}^5 using RGB values (bottom row). Varying the scale factors (columns) on the appended coordinates affects the discrete Laplace equation used to produce these soft-segmentations.

The resulting system of the colorization problem is linear: the final colors are just a weighted linear combination of each scribble’s color. In this light, we may acknowledge the connection between the colorization problem and the handle-based linear shape deformation problem, where correspondence weights are computed for each handle and each point on the surface. If we replace the Laplacian used by [Levin et al. 2004] with the cotangent Laplacian of the image surface, the resulting system would then be analogous to the Harmonic Coordinates of [Joshi et al. 2007]. The Bounded Biharmonic Weights of [Jacobson et al. 2011] show numerous advantages over Harmonic Coordinates in the realm of deformation weights. Because Bounded Biharmonic Weights also optimize an energy involving the cotangent surface Laplacian, we may similarly compute them on image surfaces and use them for colorization (see Figure 1).

One missing element is the choice of scale relationship between the Cartesian coordinates of a pixel and its appended color coordinates. For a pixel i at location $x_i, y_i \in [0, \max(w, h)]$ where w and h are the width and height measured in pixels, let its color values be $I_i, R_i, G_i, B_i, \dots \in [0, 1]$. Then we parameterize the *amount* of content-adaptiveness desired in our operators by scaling each color coordinate by a constant factor when we lift the image surface. For a pixel i in a grayscale image, the embedded coordinates are $(x_i, y_i, f_I I_i)$ and for an RGB image $(x_i, y_i, f_R R_i, f_G G_i, f_B B_i)$. The effect of tweaking these parameters is shown in Figure 2. For this example we consider Harmonic Coordinates defined on the image surface as a soft segmentation for three user scribbles. The top row shows the segmentation in pseudo-color for various scale factors, considering only the intensity channel. The bottom row considers the RGB channels (with $f_R = f_G = f_B$).

In future work, we would like to explore different color models and the ideal weighting of each color coordinate. It would also be interesting whether a similar embedding can be defined for cyclical color spaces like HSV.

References

- HERON. 60. *Metrica*. Alexandria, Roman Egypt.
- JACOBSON, A., BARAN, I., POPOVIĆ, J., AND SORKINE, O. 2011. Bounded biharmonic weights for real-time deformation. *ACM Trans. Graph.* 30, 4, 78:1–78:8.
- JOSHI, P., MEYER, M., DEROSE, T., GREEN, B., AND SANOCKI, T. 2007. Harmonic coordinates for character articulation. *ACM Trans. Graph.* 26, 3, 71:1–71:9.
- KARNI, Z., FREEDMAN, D., AND GOTSMAN, C. 2009. Energy-based image deformation. In *Proc. SGP*, 1257–1268.
- LEVIN, A., LISCHINSKI, D., AND WEISS, Y. 2004. Colorization using optimization. *ACM Trans. Graph.* 23, 689–694.
- LISCHINSKI, D., FARBMAN, Z., UYTENDAELE, M., AND SZELISKI, R. 2006. Interactive local adjustment of tonal values. *ACM Trans. Graph.* 25, 3, 646–653.
- MEYER, M., DESBRUN, M., SCHRÖDER, P., AND BARR, A. H. 2003. Discrete differential-geometry operators for triangulated 2-manifolds. In *Visualization and Mathematics III*. Springer-Verlag, 35–57.
- WARDETZKY, M., MATHUR, S., KÄLBERER, F., AND GRINSPUN, E. 2007. Discrete laplace operators: no free lunch. In *Proc. SGP*, 33–37.