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Cosserat Rods with Projective Dynamics

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1. Derivation of motion conservation momenta.

Linear momentum

Given the implicit Euler integration scheme for linear velocity:

$$\begin{aligned}\mathbf{x}^{(t+1)} &= \mathbf{x}^{(t)} + h\mathbf{v}^{(t+1)}\\ \mathbf{v}^{(t+1)} &= \mathbf{v}^{(t)} + h\mathbf{M}^{-1}(\mathbf{f}_{int(\mathbf{x}^{(t+1)})}) - \mathbf{f}_{ext}),\end{aligned}$$

the potential preserving the linear momentum is formulated as:

$$\min_{\mathbf{x}^{(t+1)}} \frac{1}{2h^2}||\mathbf{M}^{\frac{1}{2}}(\mathbf{x}^{(t+1)} - \mathbf{s}_x{}^{(t)})||_F^2,$$

where $\mathbf{s}_x{}^{(t)} = \mathbf{x}^{(t)} + h\mathbf{v}^{(t)} + h^2\mathbf{M}^{-1}\mathbf{f}_{ext}$ [BML*14].

Angular momentum

The implicit Euler integration scheme for angular velocity is defined as:

$$\begin{aligned}u^{(t+1)} &= u^{(t)} + \frac{h}{2}(u^{(t)} \circ \omega^{(t+1)}) \qquad (1)\\ \boldsymbol{\omega}^{(t+1)} &= \boldsymbol{\omega}^{(t)} + h\mathbf{J}^{-1}[\boldsymbol{\tau} - \boldsymbol{\omega}^{(t)} \times (\mathbf{J}\boldsymbol{\omega}^{(t)}),\end{aligned}$$

where the bold magnitudes denote vectors, $u^{(t)}$ and $\omega^{(t+1)}$ denote quaternions, and $\circ$ denotes a quaternion product. $\omega^{(t+1)}$ is a normalized quaternion whose imaginary part corresponds to the vector $\boldsymbol{\omega}^{(t+1)}$ and the scalar part is 0 [SM06].

Taking Eq. (1), the derivation follows:

$$u^{(t+1)} = u^{(t)} + \frac{h}{2}(u^{(t)} \circ [\boldsymbol{\omega}^{(t)} + h\mathbf{J}^{-1}[\boldsymbol{\tau} - \boldsymbol{\omega}^{(t)} \times (\mathbf{J}\boldsymbol{\omega}^{(t)})]]$$

$$u^{(t+1)} = u^{(t)} + \frac{h}{2}(u^{(t)} \circ \omega^{(t)}) + \frac{h^2}{2}u^{(t)} \circ [\mathbf{J}^{-1}[\boldsymbol{\tau} - \boldsymbol{\omega}^{(t)} \times (\mathbf{J}\boldsymbol{\omega}^{(t)})]]$$

$$0 = \frac{\mathbf{J}}{h^2}[u^{(t+1)} - u^{(t)} - \frac{h}{2}(u^{(t)} \circ \omega^{(t)}) - \frac{h^2}{2}u^{(t)} \circ [\mathbf{J}^{-1}[\boldsymbol{\tau} - \boldsymbol{\omega}^{(t)} \times (\mathbf{J}\boldsymbol{\omega}^{(t)})]]],$$

which leads to the simplified expression:

$$\frac{\mathbf{J}}{h^2}(u^{(t+1)} - s_u{}^{(t)}) = 0, \qquad (2)$$

where $s_u{}^{(t)} = u^{(t)} + \frac{h}{2}(u^{(t)} \circ \omega^{(t)}) + \frac{h^2}{2}u^{(t)} \circ [\mathbf{J}^{-1}[\boldsymbol{\tau} - \boldsymbol{\omega}^{(t)} \times (\mathbf{J}\boldsymbol{\omega}^{(t)})]]$ are the implicitly predicted orientations.

Eq. (2) is converted into an optimization problem:

$$\min_{u^{(t+1)}} \frac{1}{2h^2}||\mathbf{J}^{\frac{1}{2}}(u^{(t+1)} - s_u^{(t)})||_F^2, \qquad (3)$$

which preserves the angular momentum.

2. Insight on system convergence slowdown given the weight formulation w.r.t geometric and material parameters

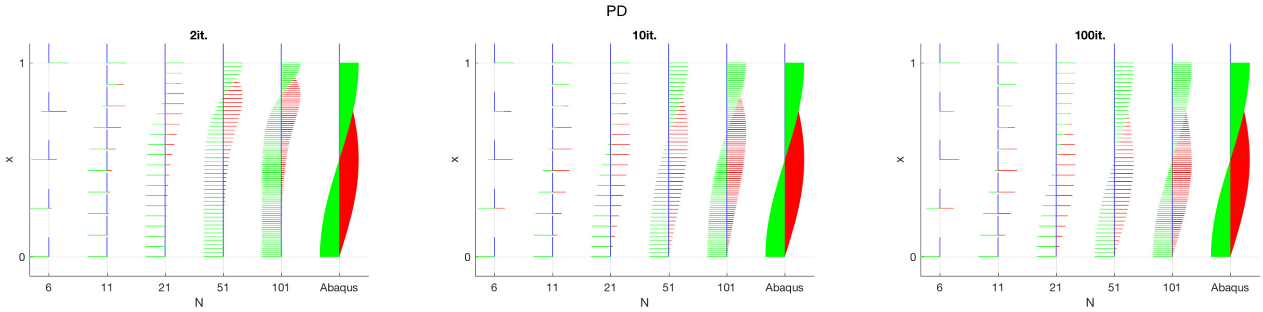
Changing the weights w_{BT} and w_{SE} has proven to affect the convergence rate of the solver. For instance, assigning a very low weight to a specific potential, makes its energy already low, making its optimization slower. However, given the Cosserat's theory continuous formulation, balancing the potentials according to material parameters is essential in order to provide realistic simulations. Therefore, as an example, we compare the convergence impact when changing the weight w_{BT} when applying a 180° twist on a 1m long rod towards a reference solution generated with a finite element method (referred to as *Abaqus*).

Figure 1a shows how our method converges by using w_{BT} in terms of geometric and material parameters. Instead of expressing the weight w_{BT} with material parameters, Fig. 1b shows the convergence rate to a reference simulation when assigning $w_{BT} = 1$, which in this case is a higher value than the one used in Fig. 1a. Given that the weight of the potential is higher, the solver converges faster to the reference solution.

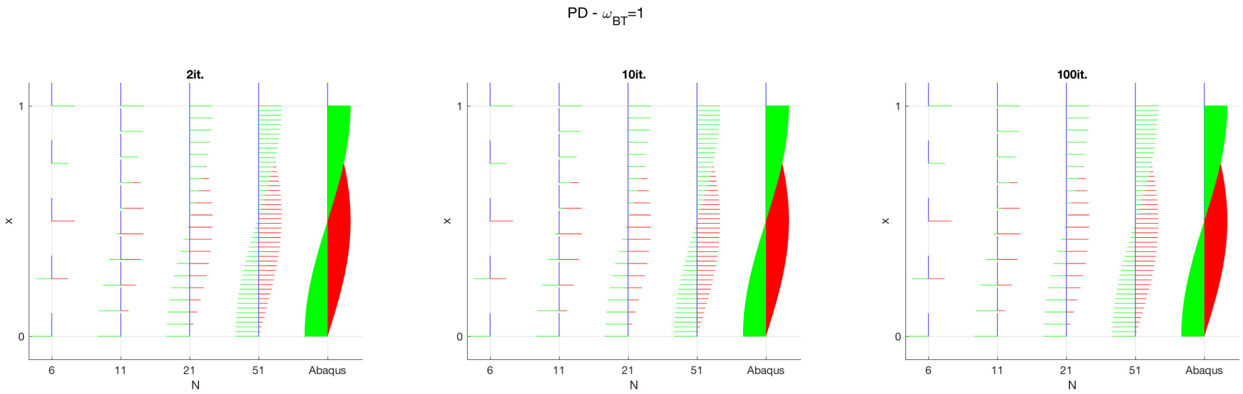
Given that this example optimizes only the twist potential, assigning a higher value to the twist potential will indeed converge faster. However, when the set up involves both the twist potential and the stretch potential, balancing them with the proposed weight formulation allows to obtain realistic simulations. In this example, 10 iterations are already enough for converging to a reference solution. As a reference, we implemented the same experiment with the state-of-the-art method [KS16]. Figure 1c shows how their implementation converges to a reference solution with already 10 iterations, and with already 2 iterations for lower resolutions.

References

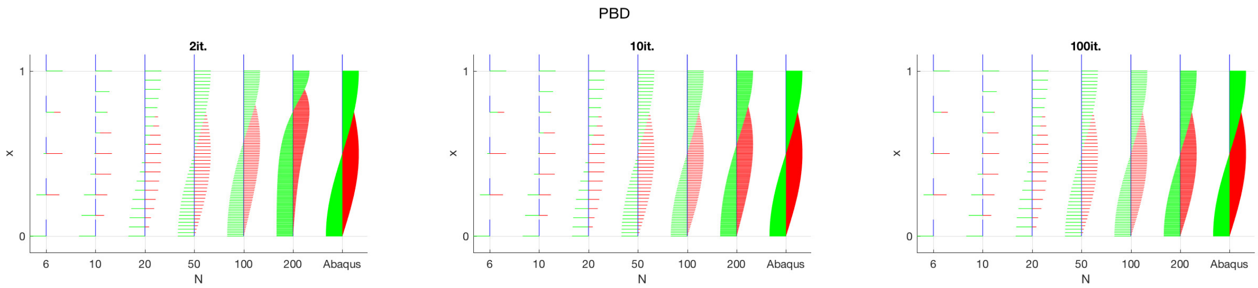
[BML*14] BOUAZIZ S., MARTIN S., LIU T., KAVAN L., PAULY M.: Projective Dynamics: Fusing Constraint Projections for Fast Simulation. *ACM Trans. Graph. (Siggraph)* (2014). 1



(a) *Cosserat rods with PD. w_{BT} defined using material and geometric parameters.*



(b) *Cosserat rods with PD. $w_{BT} = 1$.*



(c) *Cosserat rods with PBD [KS16].*

Figure 1: *Comparison of a 180° twist applied on the endpoint orientation u_{N-1} to a FEM reference solution for different mesh resolutions and number of iterations.*

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[SM06] Schwab A. L., Meijaard J.: How to Draw Euler Angles and Utilize Euler Parameters. *Proceedings of IDETC/CIE* (2006). 1