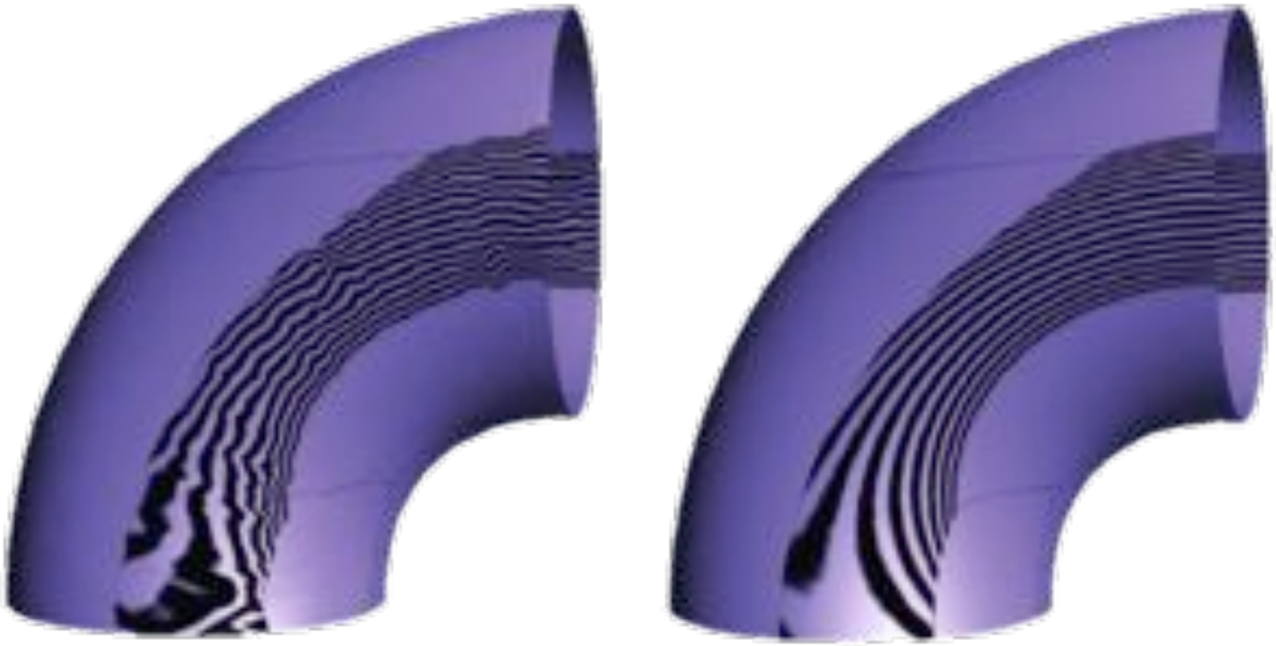
Mixed Finite Elements for Variational Surface Modeling

Alec Jacobson
Elif Tosun
Olga Sorkine
Denis Zorin

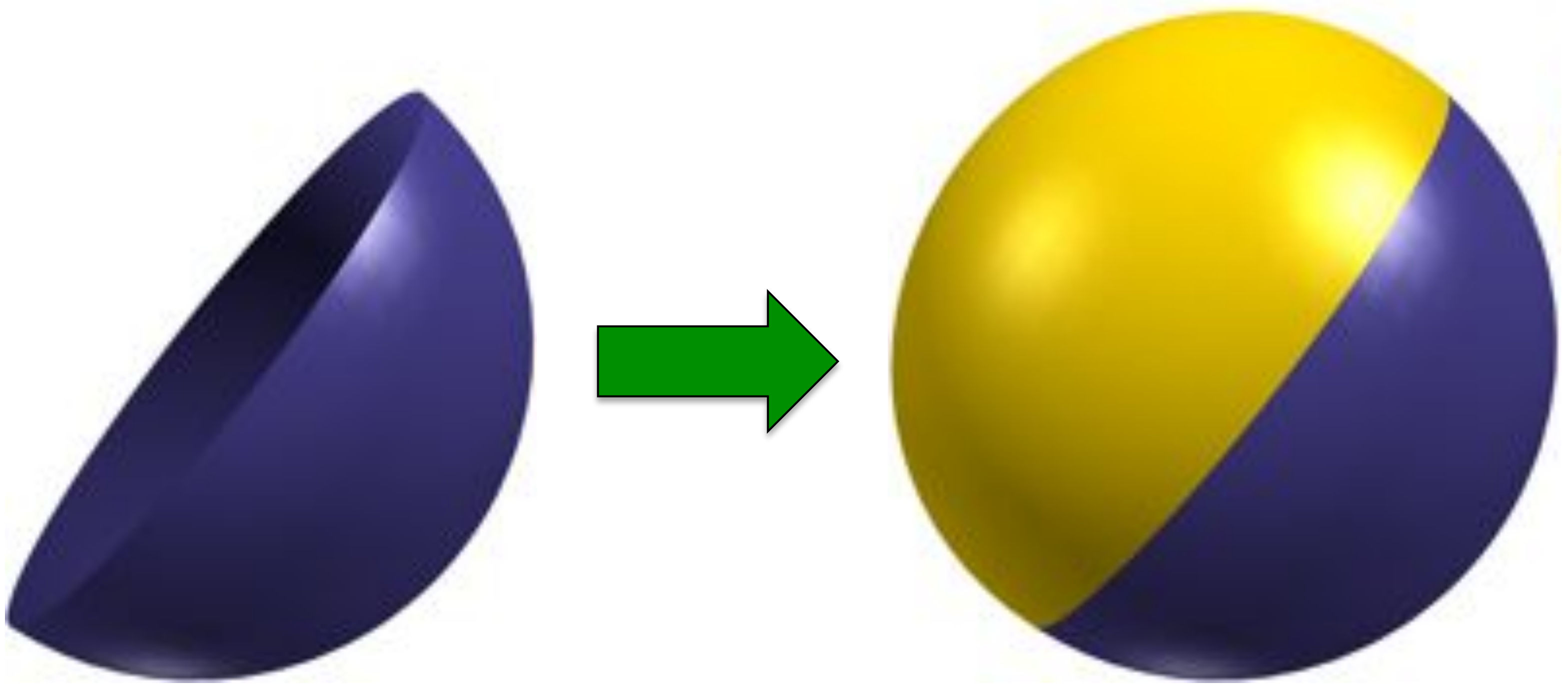
Common goal is to obtain or maintain high-quality surfaces



Low-quality surface High-quality surface

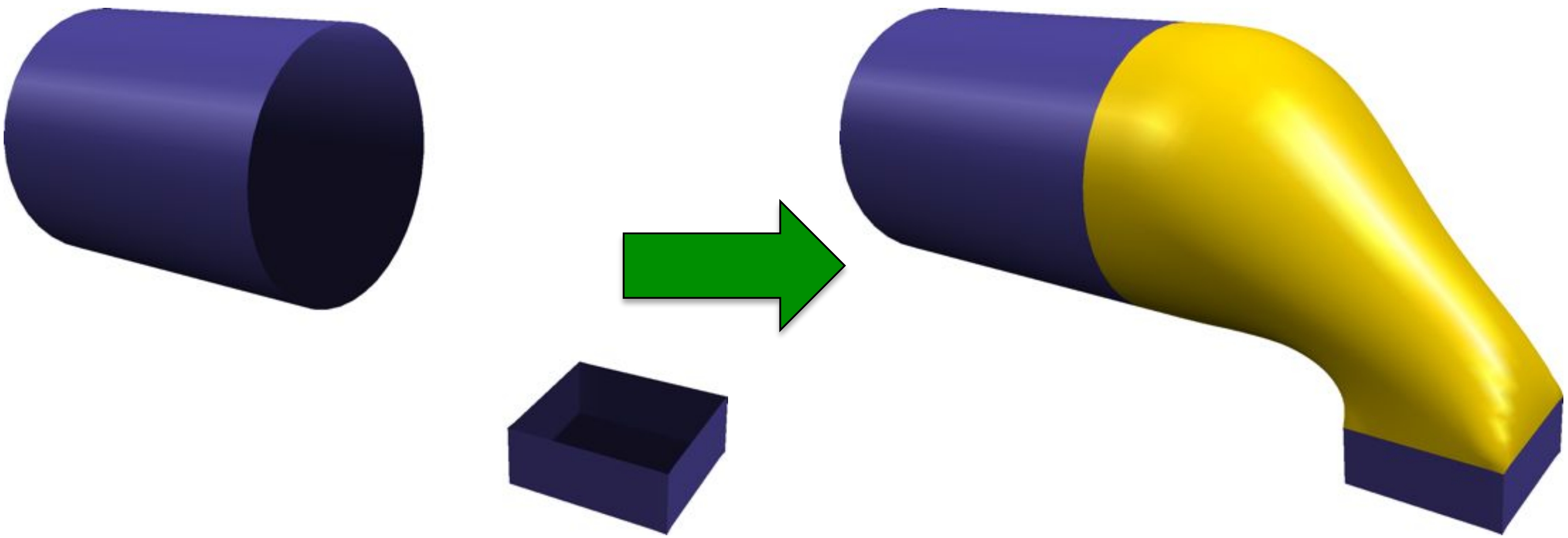
Original image: [Grinspun, Gingold, Reisman, and Zorin, 2006]

We'd like to be able to fill holes in existing surfaces, …



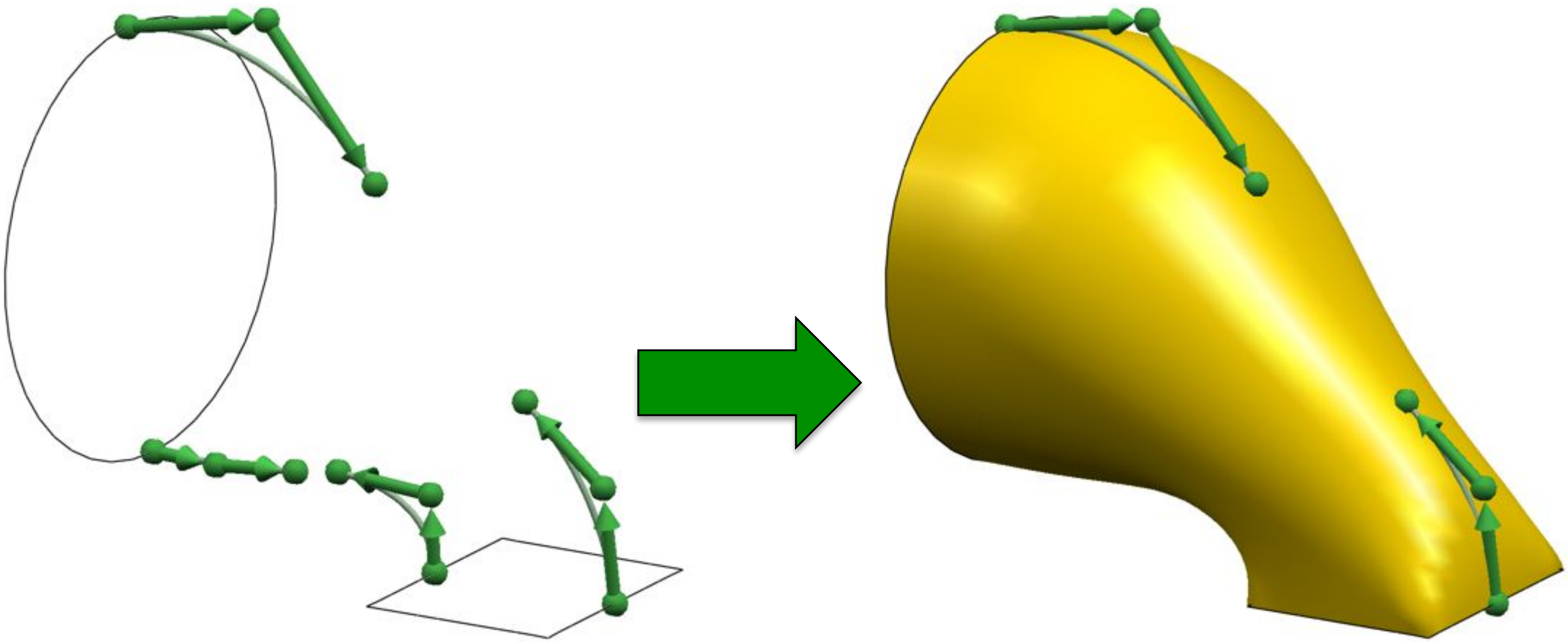
Also care about quality of boundary between new surface and old surface

… connect existing surfaces, …



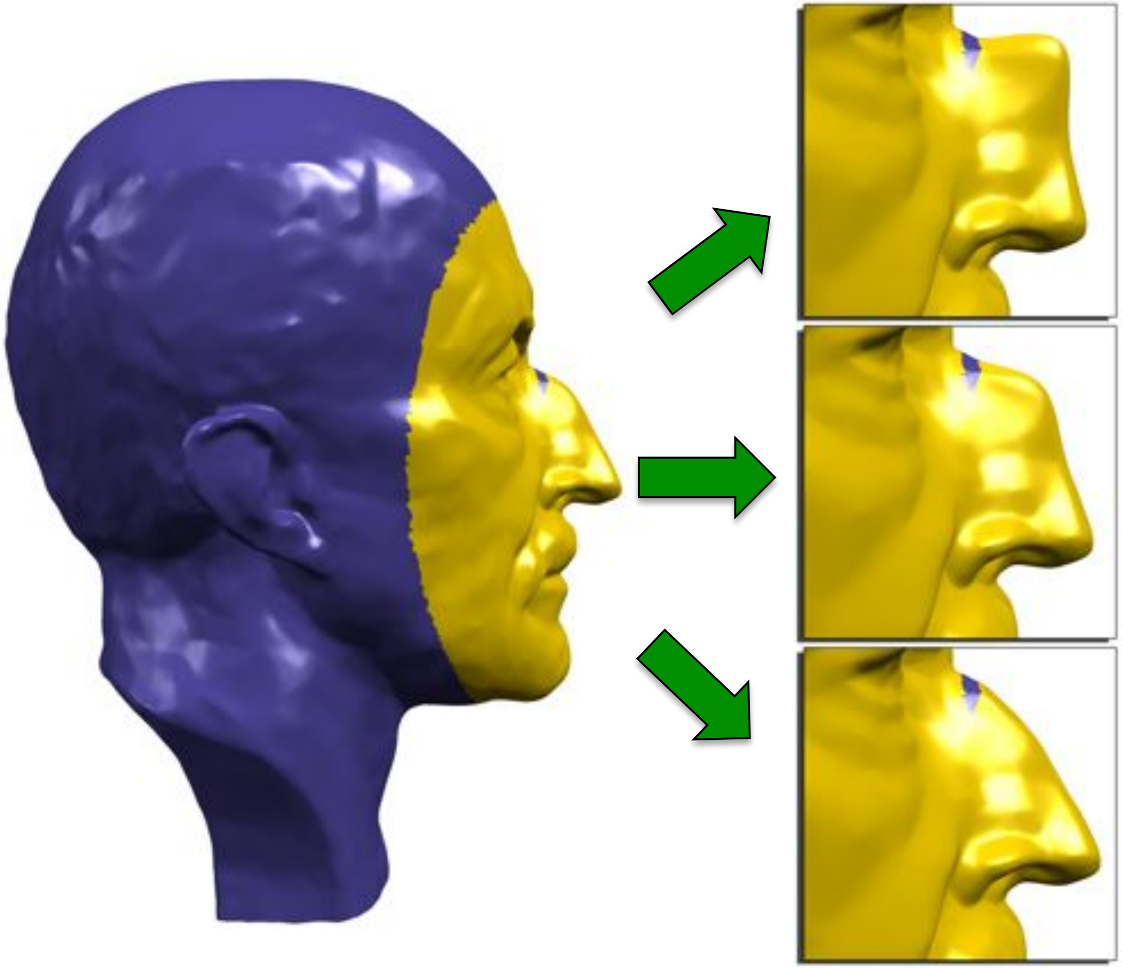
Important that boundaries of different surfaces blend smoothly

… connect existing curves, …



High-precision controls for high-quality surfaces

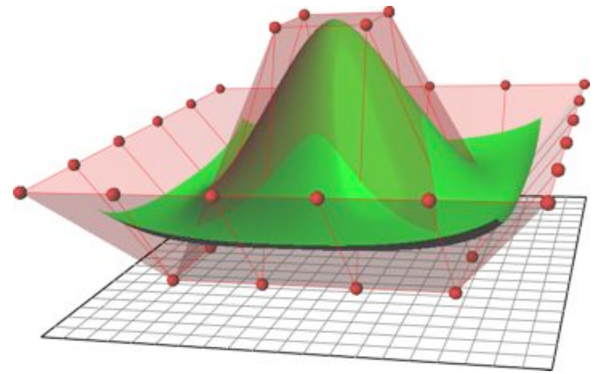
… and edit existing surfaces



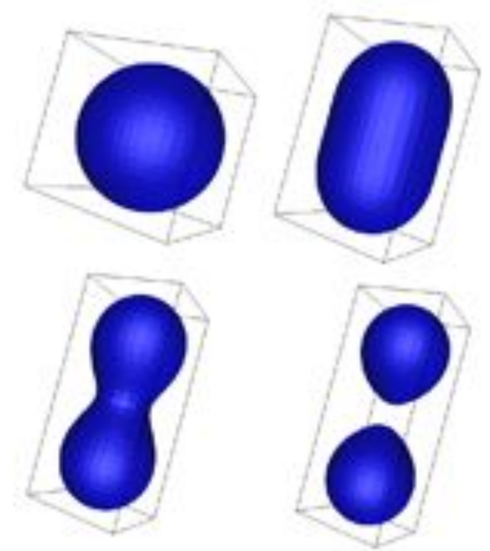
Fine-tuned edits that preserve details

There are many ways to describe high quality surfaces

NURBS



Implicit Surfaces

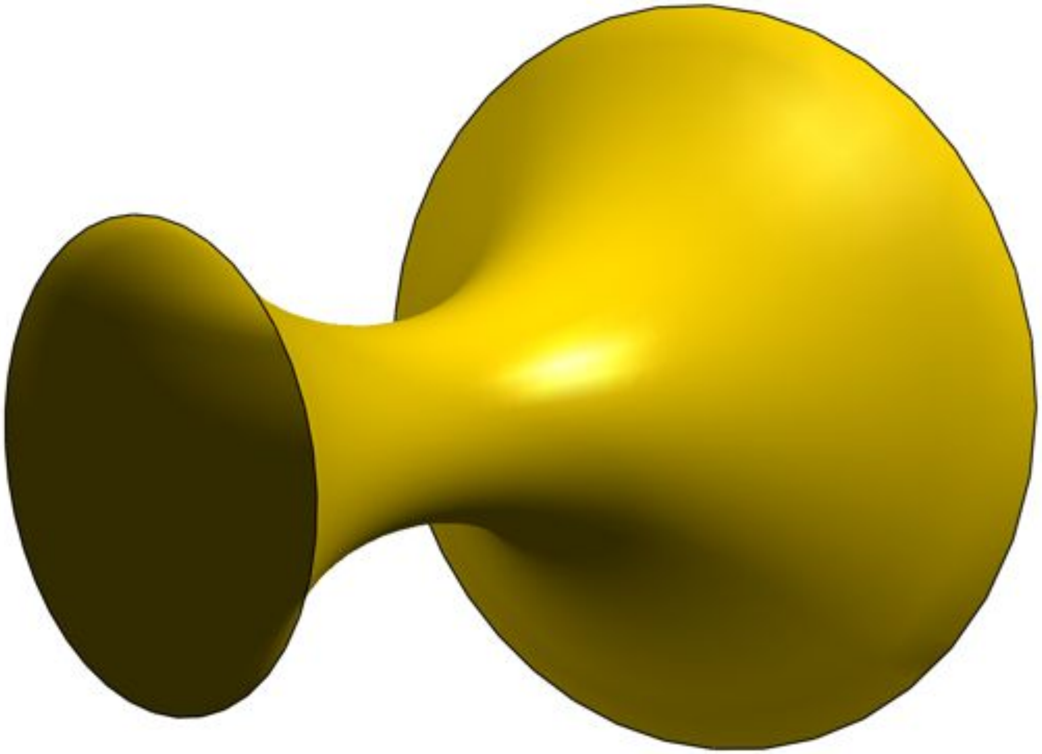


Original images: Wikipedia and [Bourke, 1997]

Solving a PDE turns surface modeling into boundary value problem

PDE captures quality we would like, e.g.:

$$\Delta \mathbf{u} = 0$$



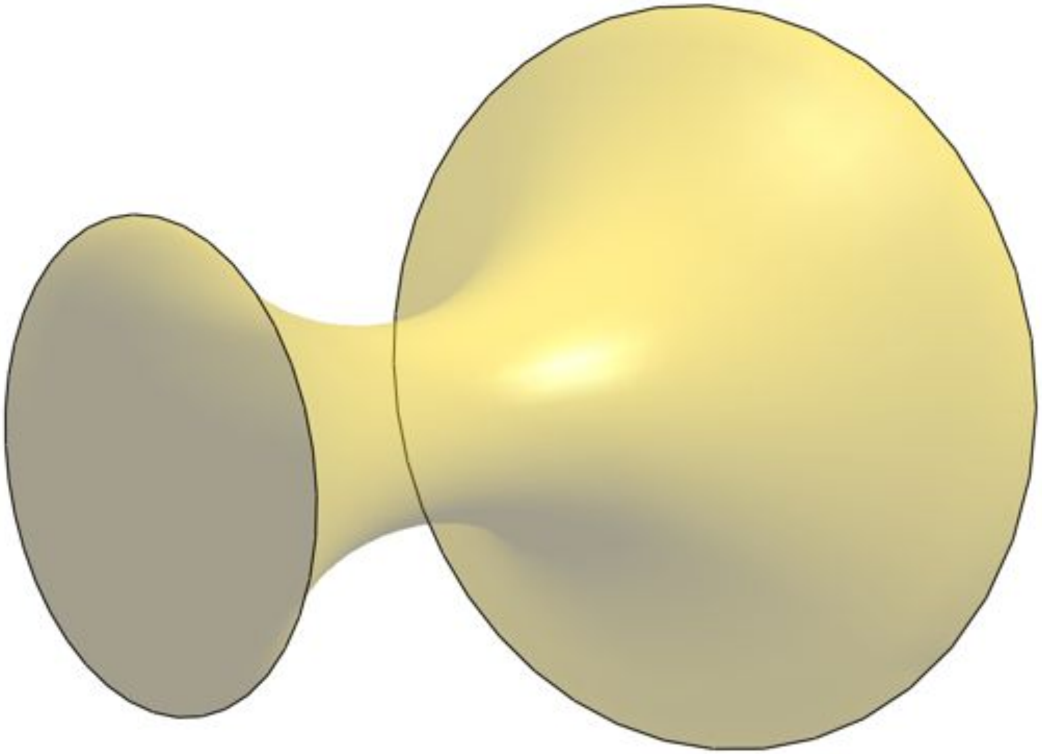
PDE surface in continuous domain

Solving a PDE turns surface modeling into boundary value problem

PDE captures quality we would like, e.g.:

$$\Delta \mathbf{u} = 0$$

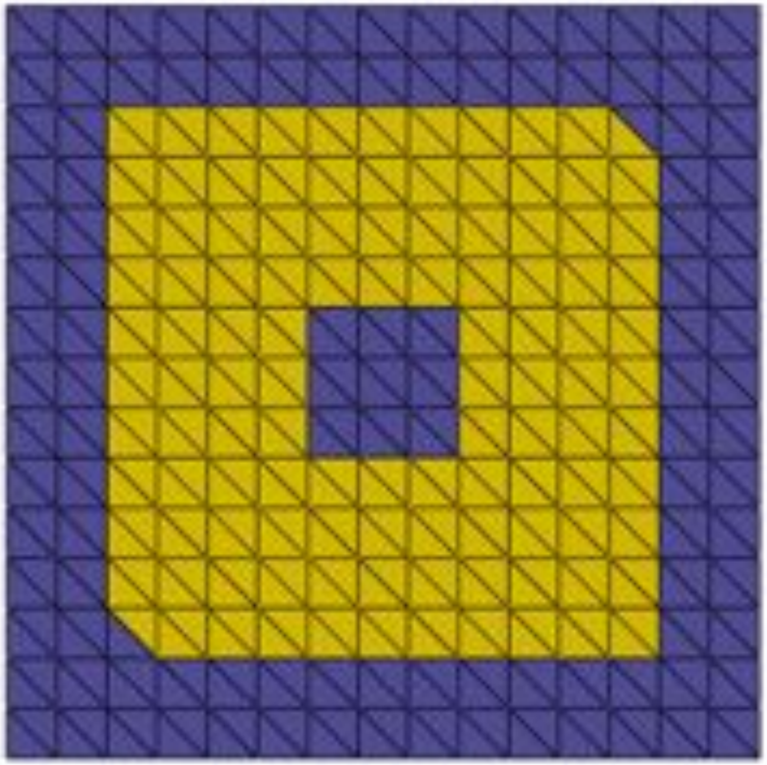
Find surface that satisfies PDE and boundary conditions



PDE surface in continuous domain

Minimizing an energy or solving a PDE can produce high quality surfaces

Discretized domain

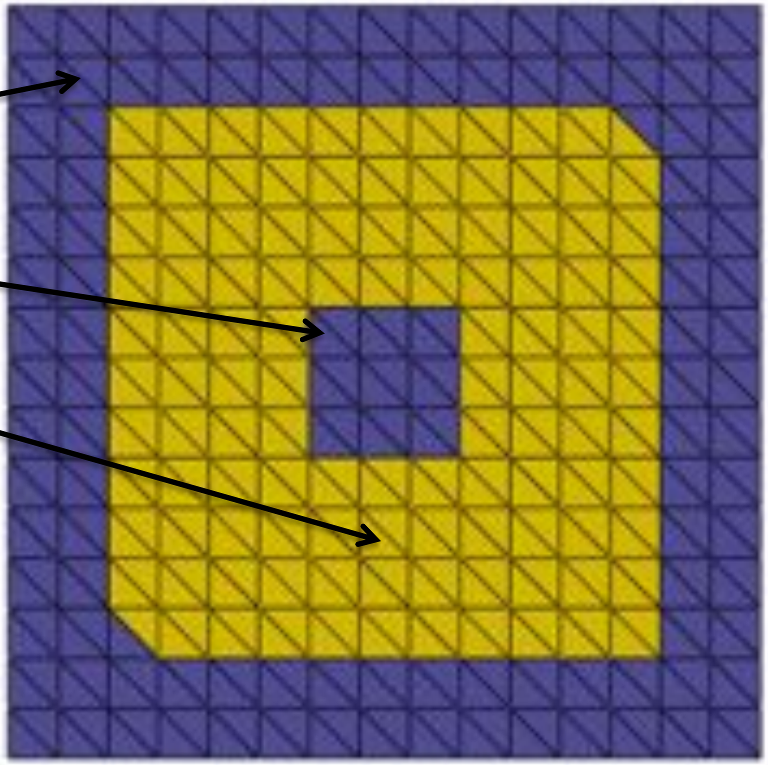


$\mathbf{u}_i$ is the (x,y,z) position of vertex i in discretization mesh

Minimizing an energy or solving a PDE can produce high quality surfaces

Discretized domain

Define exterior and interior ($\mathbf{u}_\Omega$)

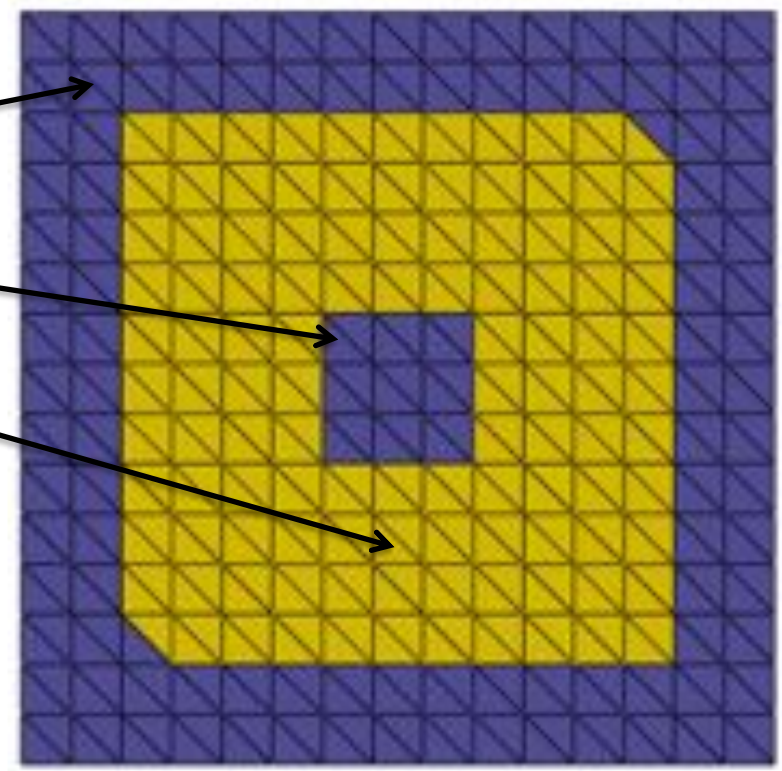


$\mathbf{u}_i$ is the (x,y,z) position of vertex i in discretization mesh

Minimizing an energy or solving a PDE can produce high quality surfaces

Discretized domain

Define exterior
and interior ($\mathbf{u}_\Omega$)



$$A \, \mathbf{u}_\Omega = b$$

Solve for $\mathbf{u}_\Omega$

$\mathbf{u}_i$ is the (x,y,z) position of vertex i in discretization mesh

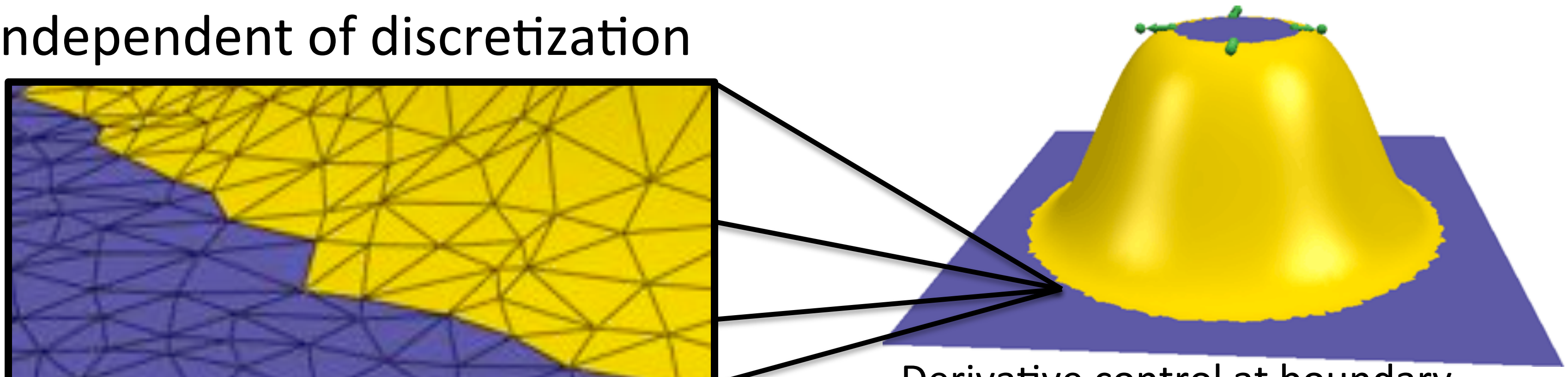
Designing a technique to discretize high order PDEs requires care

Guarantees about surface quality: interior and boundary

Expose control along boundary

Operate directly on input shape: simple triangle meshes, independent of discretization

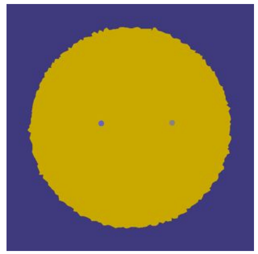
Positional control of exterior



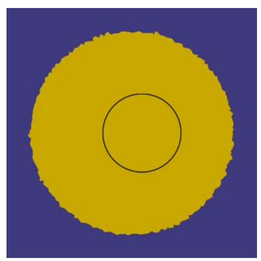
Derivative control at boundary

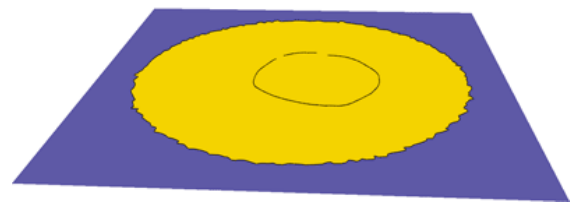
Must support different boundary types

Points

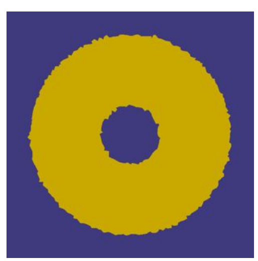


Curves





Regions



We present a technique for discretizing high order PDEs

Doesn't require high-order elements

Support points, curves and regions as boundaries

Exposes tangent and curvature control

Solution in single, sparse linear solve

We present a technique for discretizing high order PDEs

Doesn't require high-order elements

Support points, curves and regions as boundaries

Exposes derivative and curvature control

Solution in single, sparse linear solve

Real-time modeling and deformation

Convergence for high order PDEs

Biharmonic and Triharmonic equations serve as running examples

Biharmonic equation

$$\Delta^2 \mathbf{u} = 0$$

Notation:

$$\Delta \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = \mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{u}_{zz}$$

$$\Delta^k \mathbf{u} = \Delta(\Delta^{k-1} \mathbf{u})$$

Biharmonic and Triharmonic equations serve as running examples

Biharmonic equation

$$\Delta^2 \mathbf{u} = 0$$

Laplacian energy

$$E_B = \frac{1}{2}\langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min$$

Notation:

$$\Delta \mathbf{u} = \nabla \cdot \nabla \, \mathbf{u} = \mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{u}_{zz}$$

$$\Delta^k \mathbf{u} = \Delta(\Delta^{k-1}\mathbf{u})$$

$$\langle f, g \rangle_\Omega = \int_\Omega fg$$

Biharmonic and Triharmonic equations serve as running examples

Biharmonic equation

$$\Delta^2 \mathbf{u} = 0$$

Laplacian energy

$$E_B = \frac{1}{2}\langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min$$

Triharmonic equation

$$\Delta^3 \mathbf{u} = 0$$

Notation:

$$\Delta \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = \mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{u}_{zz}$$

$$\Delta^k \mathbf{u} = \Delta(\Delta^{k-1}\mathbf{u})$$

$$\langle f, g \rangle_\Omega = \int_\Omega fg$$

Biharmonic and Triharmonic equations serve as running examples

Biharmonic equation

$$\Delta^2 \mathbf{u} = 0$$

Laplacian energy

$$E_B = \frac{1}{2}\langle \Delta\mathbf{u}, \Delta\mathbf{u}\rangle_{\Omega_0} \rightarrow \min$$

Triharmonic equation

$$\Delta^3 \mathbf{u} = 0$$

Laplacian gradient energy

$$E_T = \frac{1}{2}\langle \nabla\Delta\mathbf{u}, \nabla\Delta\mathbf{u}\rangle_{\Omega_0} \rightarrow \min$$

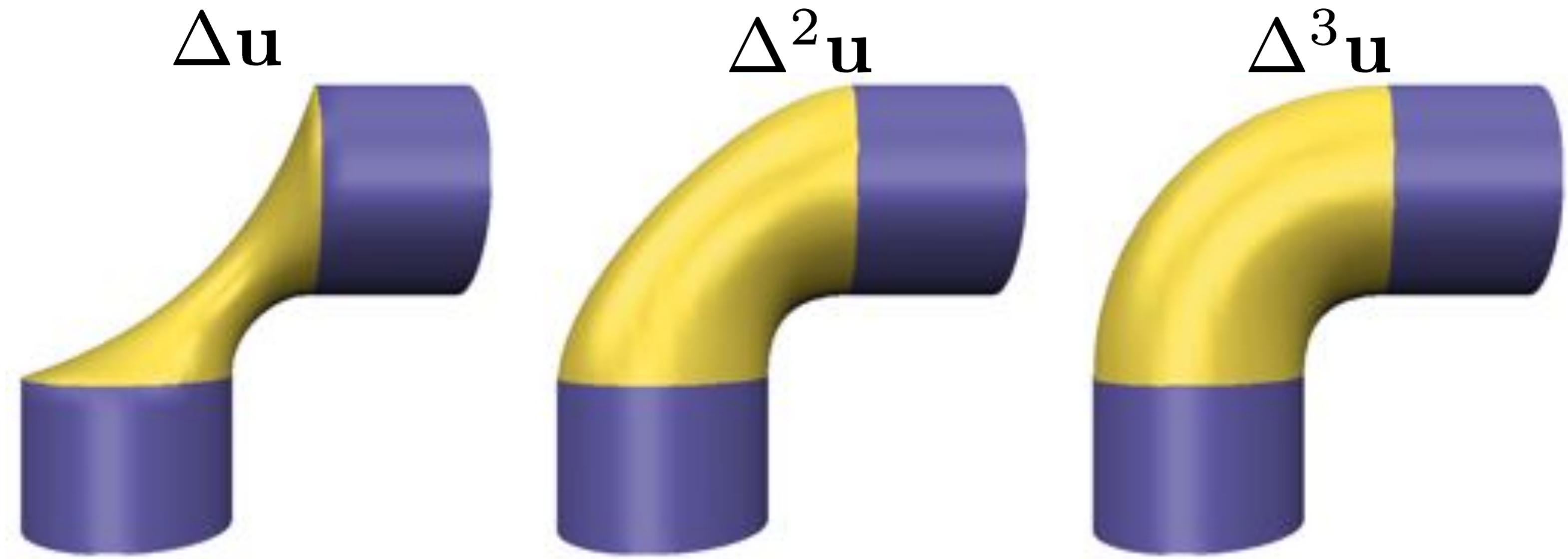
Notation:

$$\Delta\mathbf{u} = \nabla\cdot\nabla\,\mathbf{u} = \mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{u}_{zz}$$

$$\Delta^k\mathbf{u} = \Delta(\Delta^{k-1}\mathbf{u})$$

$$\langle f, g\rangle_\Omega = \int_\Omega fg$$

Bi-/Tri- harmonic equations produce smooth surfaces and boundaries



Soap film
C^0 at boundary
Positional control at boundary

Thin plate
C^1 at boundary
+Tangent control at boundary

Curvature variation minimizing
C^2 at boundary
+Curvature control at boundary

Original image: [Botsch and Kobbelt, 2004]

Previous works have limitations

Simple domains, analytic boundaries

[Bloor and Wilson 1990]

Model shaped minimization of curvature variation energy

[Moreton and Séquin 1992]

Interpolate curve networks, local quadratic fits and finite differences

[Welch and Witkin 1994]

Uniform-weight discrete Laplacian

[Taubin 1995]

Cotangent-weight discrete Laplacian

[Pinkall and Polthier 1993],
[Wardetzky et al. 2007],
[Reuter et al. 2009]

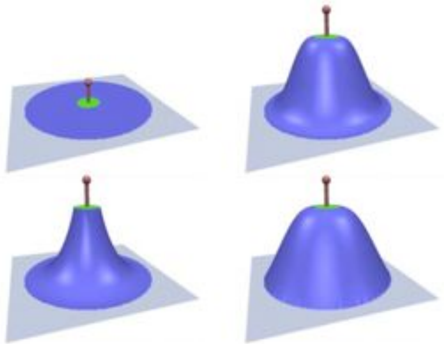
We can show previous solutions are applications of mixed FEM approach

[Clarenz et al., 2004]

- Willmore Flow (fourth-order PDE)
- Positions and co-normals on boundary

[Botsch and Kobbelt, 2004]

- Discretization of k-harmonic equations



Discrete boundary conditions found in these can be derived from continuous case

Standard finite element method would require high-order elements

Need many more degrees of freedom

Existing high-order representations are neither practical, nor popular

Need low order, C^0, workarounds:

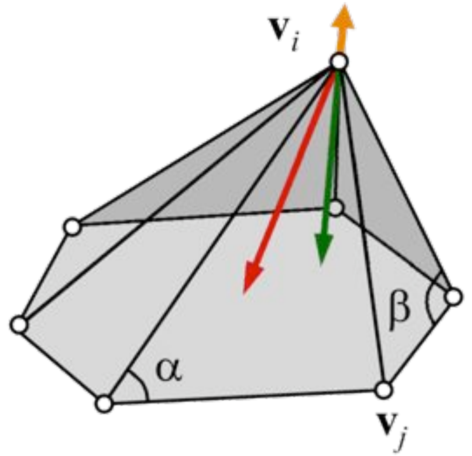
e.g. mixed FEM

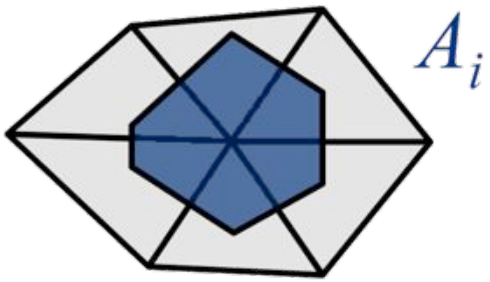
Discrete Geometric Discretization not easily connected to continuous case

Idea is to define mesh analog of continuous geometric quantity

E.g. Laplace-Beltrami operator integrated over vertex area

Used often in geometric modeling





We introduce mixed finite elements for variational surface modeling

Introduce new variable to convert high-order problem into two low-order problems

$$\Delta^2 \mathbf{u} = 0$$

$$\Downarrow$$

Solve two problems simultaneously

$$\Delta \mathbf{u} = \mathbf{v}$$
$$\Delta \mathbf{v} = 0$$

We introduce mixed finite elements for variational surface modeling

Introduce new variable to convert high-order problem into two low-order problems

$$\frac{1}{2}\langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \to \min$$

$$\Downarrow$$

Solve two problems simultaneously

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} \to \min, \quad \text{s.t. } \Delta \mathbf{u} = \mathbf{v}$$

New variable needs to be enforced as hard constraint

We use Lagrange multipliers to enforce the new variable

$$\frac{1}{2}\langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min$$

We use Lagrange multipliers to enforce the new variable

$$\frac{1}{2}\langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \to \min$$

(New variable: $\mathbf{v}$)

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} \to \min, \text{ s.t. } \Delta \mathbf{u} = \mathbf{v}$$

We use Lagrange multipliers to enforce the new variable

$$\frac{1}{2}\langle \Delta\mathbf{u}, \Delta\mathbf{u}\rangle_{\Omega_0} \rightarrow \min$$

(New variable: $\mathbf{v}$)

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \boldsymbol{\Delta}\mathbf{u} = \mathbf{v}$$

(Lagrange multiplier: λ)

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \boldsymbol{\Delta}\mathbf{u} - \mathbf{v}\rangle_{\Omega_0} \rightarrow \min$$

We use Lagrange multipliers to enforce the new variable

$$\frac{1}{2}\langle \Delta\mathbf{u}, \Delta\mathbf{u}\rangle_{\Omega_0} \rightarrow \min$$

(New variable: $\mathbf{v}$)

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \Delta\mathbf{u} = \mathbf{v}$$

(Lagrange multiplier: λ)

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \Delta\mathbf{u} - \mathbf{v}\rangle_{\Omega_0} \rightarrow \min$$

(Green's Identity)

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n}\rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u}\rangle_{\Omega_0} \rightarrow \min$$

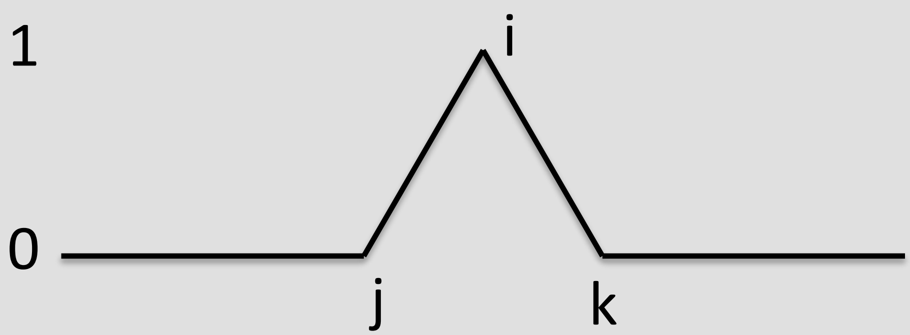
Discretize each variable using piece-wise linear elements

$$\mathbf{u} = \sum_{i \in \Omega} u_i \phi_i$$

$$\mathbf{v} = \sum_{i \in \Omega} v_i \phi_i$$

$$\lambda = \sum_{i \in \Omega} \lambda_i \phi_i$$

Hat function: ϕ_i



1 at vertex i, 0 at all other vertices

Linearly interpolated across edges, faces of mesh

Take derivatives of energy to find minimum

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n}\rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u}\rangle_{\Omega_0} \to \min$$

With respect to $\mathbf{v}$:

$$\sum_{j\in I}(\mathbf{v}_j - \lambda_j)\langle \phi_j, \phi_i\rangle_{\Omega_0} = 0$$

Take derivatives of energy to find minimum

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \tfrac{\partial \mathbf{u}}{\partial n}\rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u}\rangle_{\Omega_0} \to \min$$

With respect to $\mathbf{v}$:

$$\sum_{j\in I}(\mathbf{v}_j - \lambda_j)\langle \phi_j, \phi_i\rangle_{\Omega_0} = 0$$

With respect to $\mathbf{u}$:

$$-\sum_{j\in I}\lambda_j\langle \nabla\phi_j, \nabla\phi_i\rangle_{\Omega_0} = 0$$

Take derivatives of energy to find minimum

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n}\rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u}\rangle_{\Omega_0} \to \min$$

With respect to $\mathbf{v}$:

$$\sum_{j\in I}(\mathbf{v}_j - \lambda_j)\langle \phi_j, \phi_i\rangle_{\Omega_0} = 0$$

With respect to $\mathbf{u}$:

$$-\sum_{j\in I}\lambda_j\langle \nabla\phi_j, \nabla\phi_i\rangle_{\Omega_0} = 0$$

With respect to λ:

$$\sum_{j\in I}\mathbf{v}_j\langle \phi_j, \phi_i\rangle_{\Omega_0} + \sum_{j\in I}\frac{\partial \mathbf{u}_j}{\partial n}\langle \phi_j, \phi_i\rangle_{\partial\Omega_0} - \sum_{j\in I}\mathbf{u}_j\langle \nabla\phi_j, \nabla\phi_i\rangle_{\Omega_0} = 0$$

Take derivatives of energy to find minimum

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n}\rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u}\rangle_{\Omega_0} \to \min$$

With respect to $\mathbf{v}$:

$$\sum_{j\in I}(\mathbf{v}_j - \lambda_j)\langle \phi_j, \phi_i\rangle_{\Omega_0} = 0$$

With respect to $\mathbf{u}$:

$$-\sum_{j\in I}\lambda_j\langle \nabla\phi_j, \nabla\phi_i\rangle_{\Omega_0} = 0$$

With respect to λ:

$$\sum_{j\in I}\mathbf{v}_j\langle \phi_j, \phi_i\rangle_{\Omega_0} + \sum_{j\in I}\frac{\partial \mathbf{u}_j}{\partial n}\langle \phi_j, \phi_i\rangle_{\partial\Omega_0} - \sum_{j\in I}\mathbf{u}_j\langle \nabla\phi_j, \nabla\phi_i\rangle_{\Omega_0} = 0$$

Take derivatives of energy to find minimum

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n} \rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u} \rangle_{\Omega_0} \to \min$$

With respect to $\mathbf{v}$:

Lagrange multiplier has disappeared

$$\sum_{j \in I} \mathbf{v}_j \langle \nabla\phi_j, \nabla\phi_i \rangle_{\Omega_0} = 0$$

With respect to λ:

$$\sum_{j \in I} \mathbf{v}_j \langle \phi_j, \phi_i \rangle_{\Omega_0} + \sum_{j \in I} \frac{\partial \mathbf{u}_j}{\partial n} \langle \phi_j, \phi_i \rangle_{\partial\Omega_0} - \sum_{j \in I} \mathbf{u}_j \langle \nabla\phi_j, \nabla\phi_i \rangle_{\Omega_0} = 0$$

Solve simultaneously as one big system

Move known parts to right-hand side

Rewrite in block matrix form:

$$\begin{bmatrix} -M & L \\ L & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} -L\bar{\mathbf{u}} - N\bar{\mathbf{n}} \\ 0 \end{bmatrix}$$

Discrete Laplacian
$L_{ij} = \langle \nabla\phi_i, \nabla\phi_j \rangle$

Mass matrix
$M_{ij} = \langle \phi_i, \phi_j \rangle$

Neumann matrix
$N_{ij} = \langle \phi_i, \phi_j \rangle_{\partial\Omega}$

Solve simultaneously as one big system

Move known terms to right-hand side

Rewrite in block matrix form:

$$A \, \mathbf{u}_{\Omega} = b$$

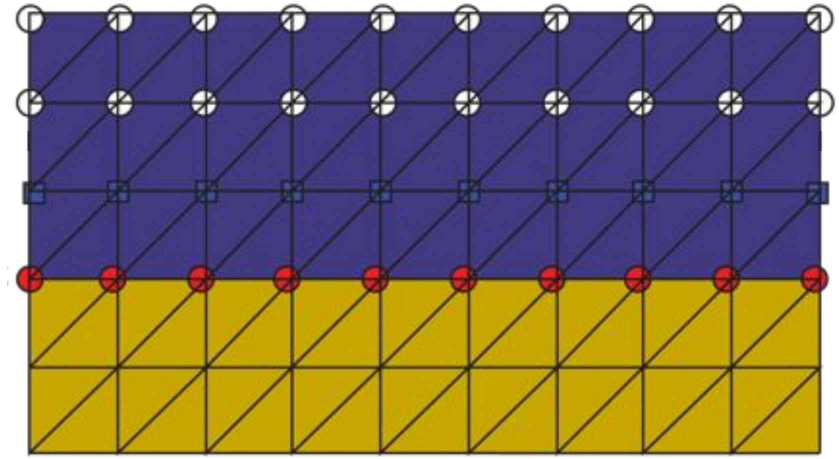
We can solve deformations in real-time using pre-factored matrix

Point boundaries

Curve boundaries with derivatives

Region boundaries are also derived from continuous case

Use two rings of boundary instead of one ring with specified derivatives



Resulting systems are similar, different right-hand sides

Helps simplify implementation to support all three boundary types

Triharmonic offers more boundary control, better smoothness

Introduce two new variables to convert high-order problem into three low-order problems

$$\Delta^3 \mathbf{u} = 0$$

$$\Downarrow$$

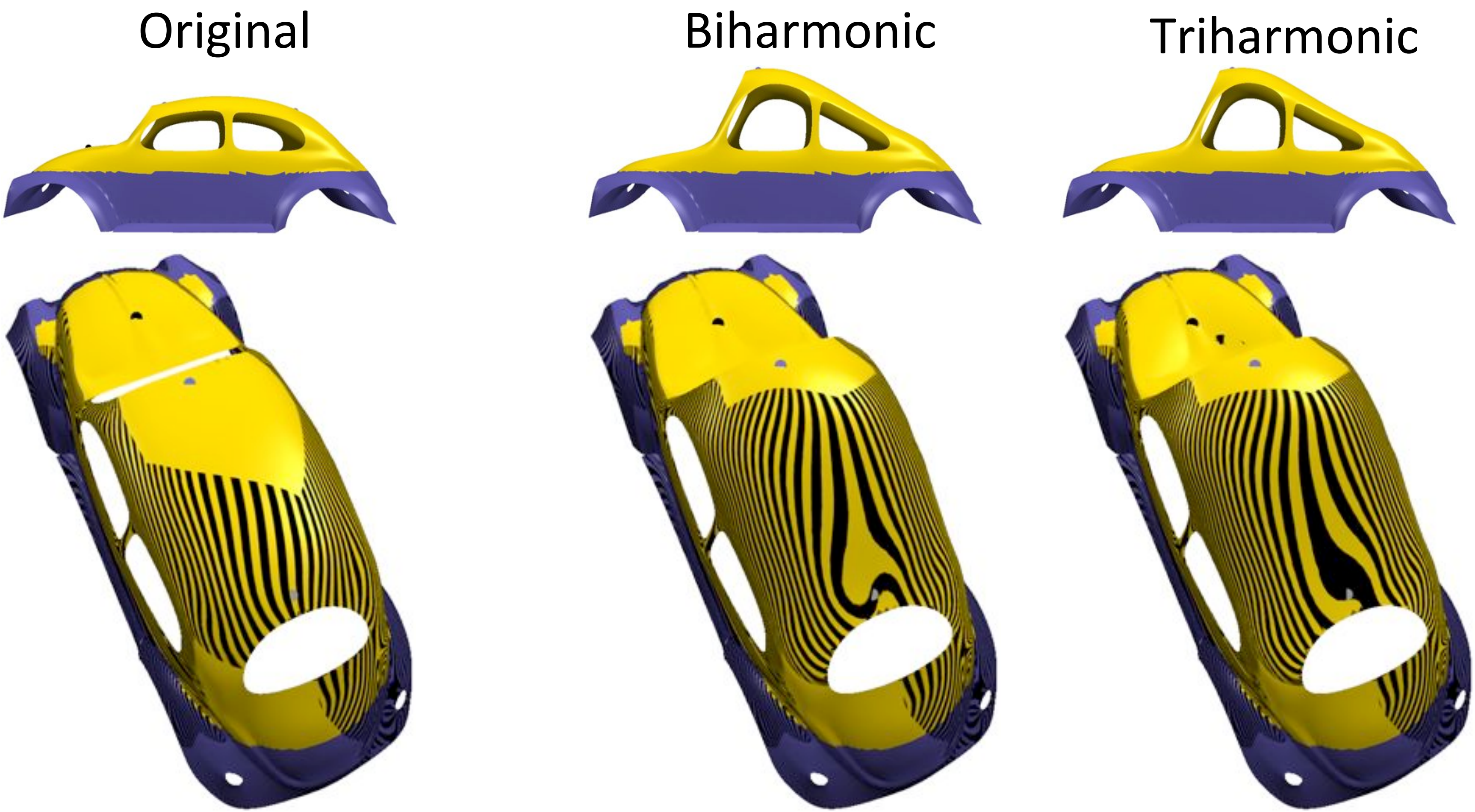
$$\Delta \mathbf{u} = \mathbf{v}$$
$$\Delta \mathbf{v} = \mathbf{w}$$
$$\Delta \mathbf{w} = 0$$

Solve three problems simultaneously

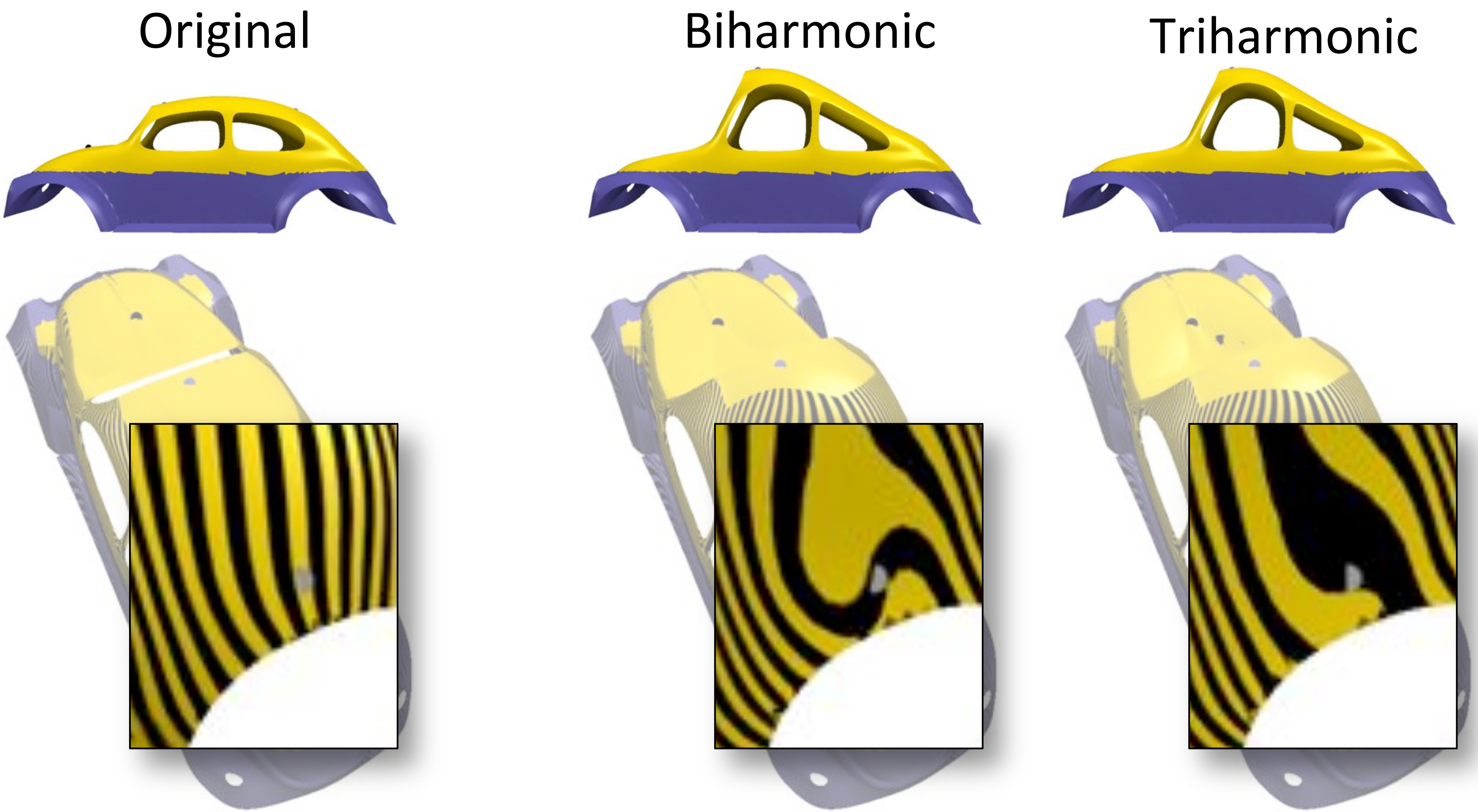
Need even more Langrange multipliers

But in the end we get a structurally similar, linear system

Triharmonic guarantees C^2 continuity at boundaries



Triharmonic guarantees C^2 continuity at boundaries



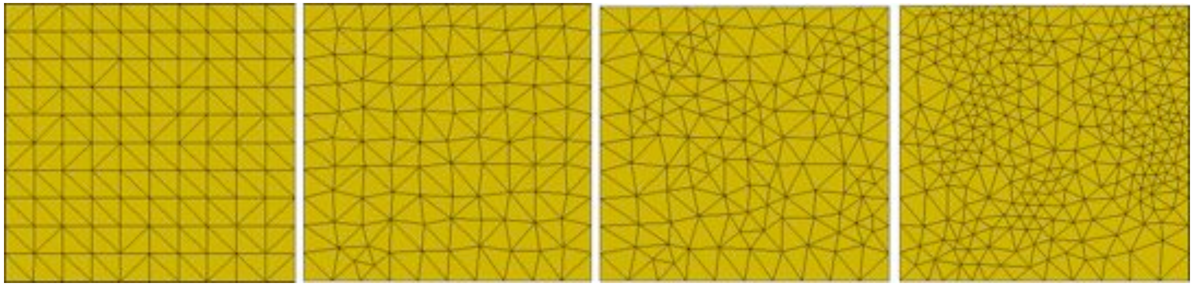
Convergence with refinement is also guaranteed and mesh independent

Test solved functions
against
known analytic functions

$$\Delta^k \mathbf{u} = \Delta^k \mathbf{u}_t$$

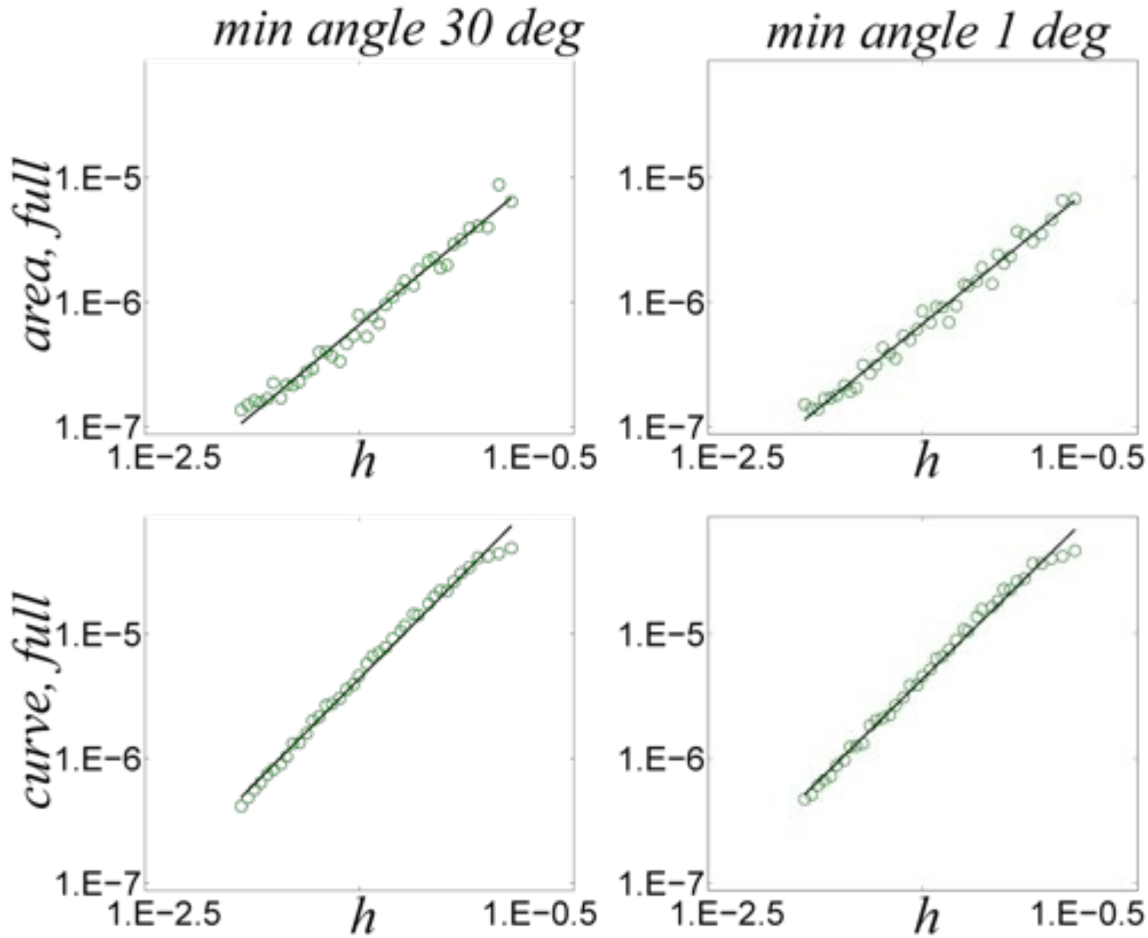
$$\text{error} = \|\mathbf{u} - \mathbf{u}_t\|$$

Over varying mesh resolution and irregularity

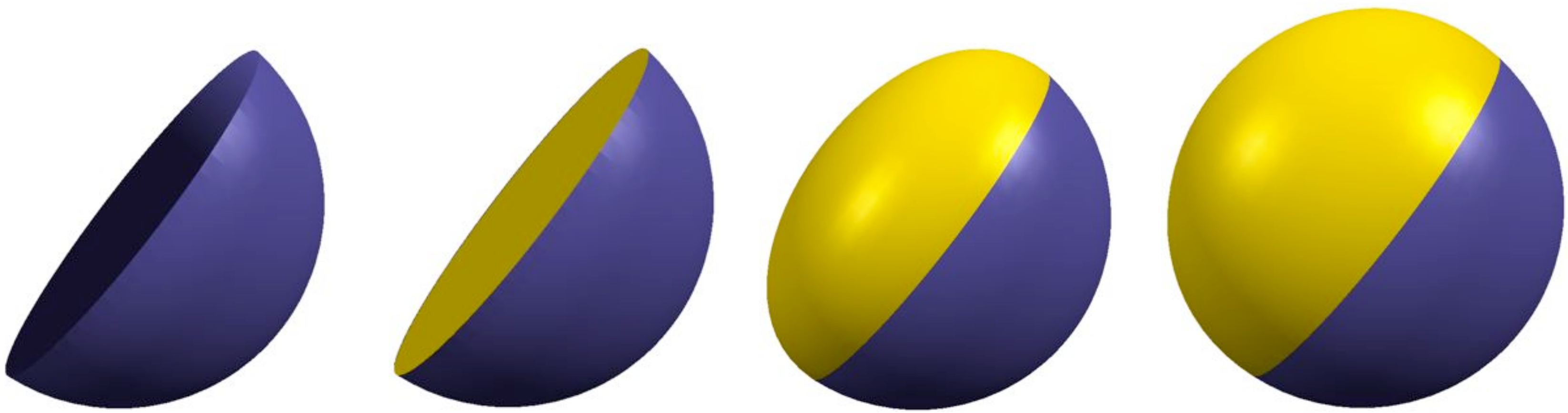


Observe nearly optimal convergence for biharmonic

Boundary types don't have affect on convergence

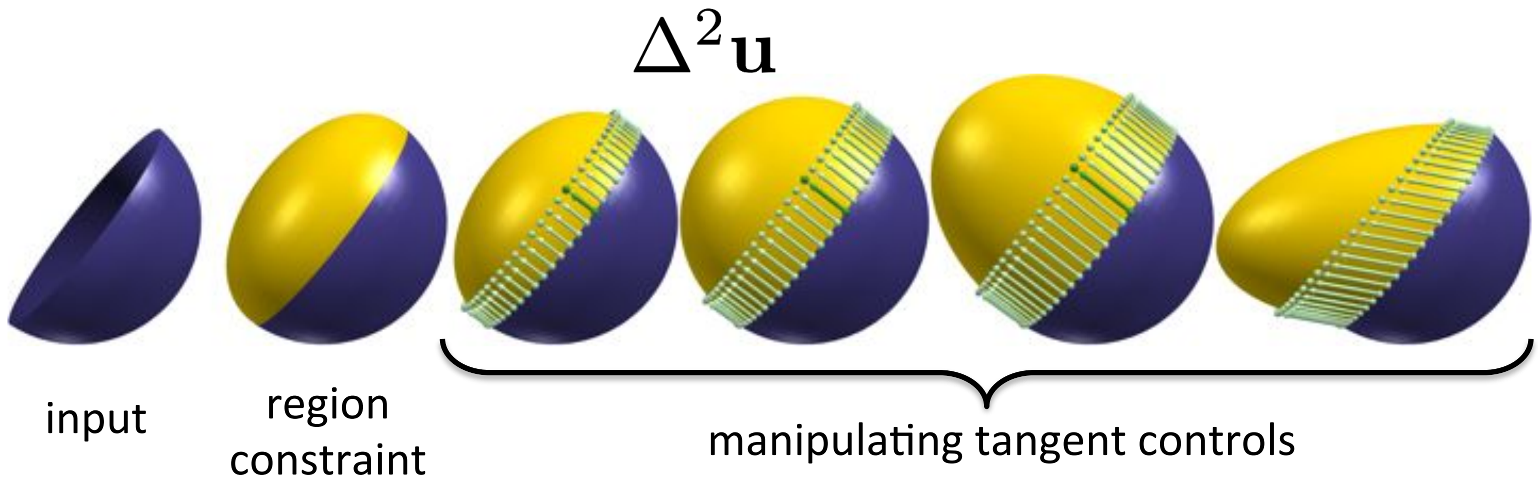


High-order PDEs are more suitable for completing surfaces



original $\qquad \Delta \mathbf{u} \qquad \Delta^2 \mathbf{u} \qquad \Delta^3 \mathbf{u}$

High-order PDEs are more suitable for completing surfaces

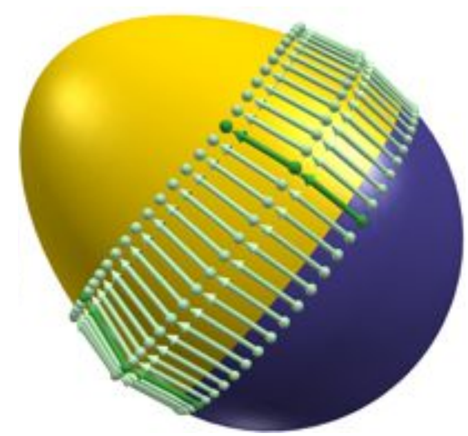


High-order PDEs are more suitable for completing surfaces

$$\Delta^3 \mathbf{u}$$

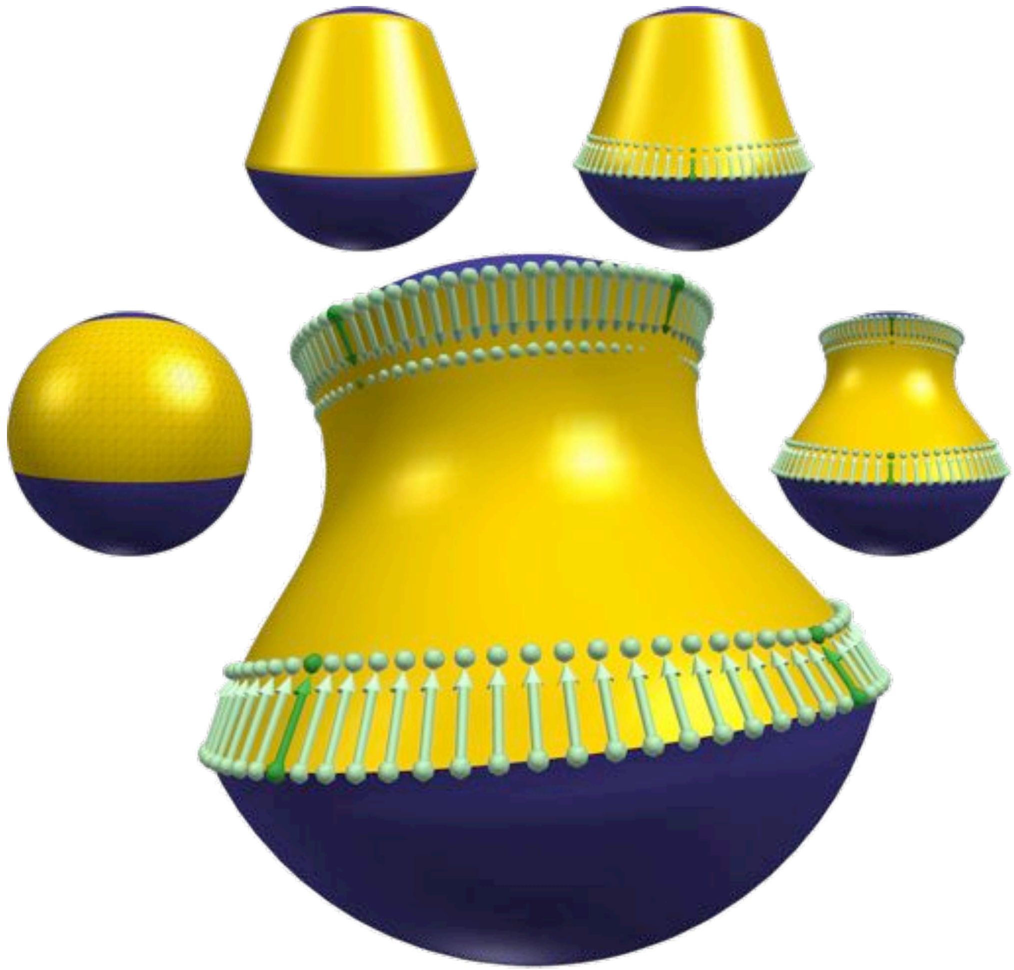
input

region constraint

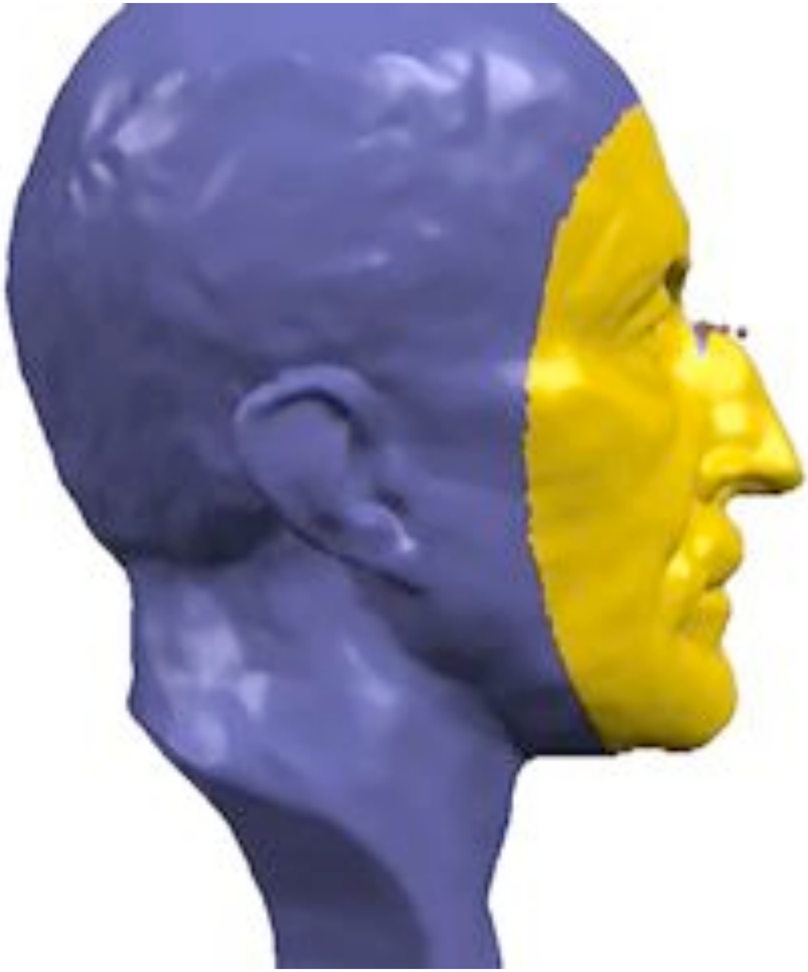


manipulating curvature controls

Specifying derivatives adds greater control to shape manipulation



Specifying curvatures adds even greater control to shape manipulation



Curve boundaries well suited for draw-and-drag manipulation

We provide a discretization technique for high-order energies or PDEs

Reduce to low order using new constrained variables

Use same constraint structure to enforce region conditions

Convergence high-order PDEs, with discretization independence

We provide a discretization technique for high-order energies or PDEs

Reduce to low order using new constrained variables

Use same constraint structure to enforce region conditions

Convergence high-order PDEs, with discretization independence

Future Work:

Improve convergence of triharmonic solution

Effect of non-flat metric

We gratefully acknowledge our funding

This work was supported in part by an NSF award IIS- 0905502.

Mixed Finite Elements for Variational Surface Modeling

Alec Jacobson (jacobson@cs.nyu.edu)
Elif Tosun
Olga Sorkine
Denis Zorin