



# Mixed Finite Elements for Variational Surface Modeling

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# Motivation

Produce high-quality surfaces

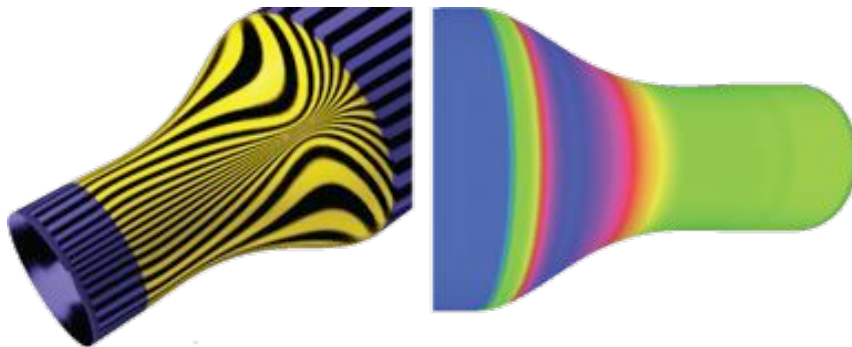
Via energy minimization

Or solving Partial Differential Equations

$$\langle f, g \rangle = \int f g$$

Laplacian energy

$$E_B = \frac{1}{2} \langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min$$



Biharmonic equation

$$\Delta^2 \mathbf{u} = 0$$

Laplacian gradient energy

$$E_T = \frac{1}{2} \langle \nabla \Delta \mathbf{u}, \nabla \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min$$

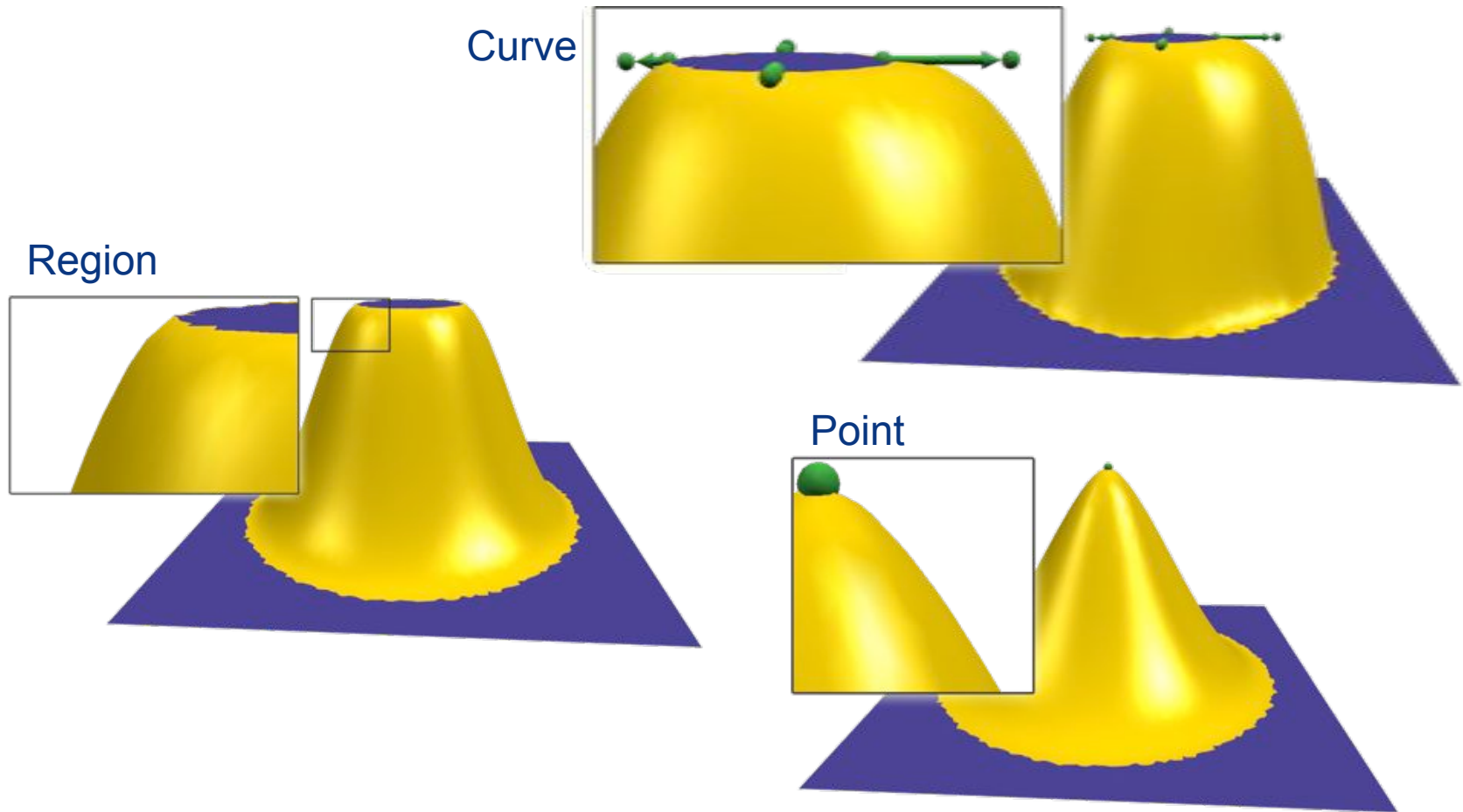


Triharmonic equation

$$\Delta^3 \mathbf{u} = 0$$

# Motivation

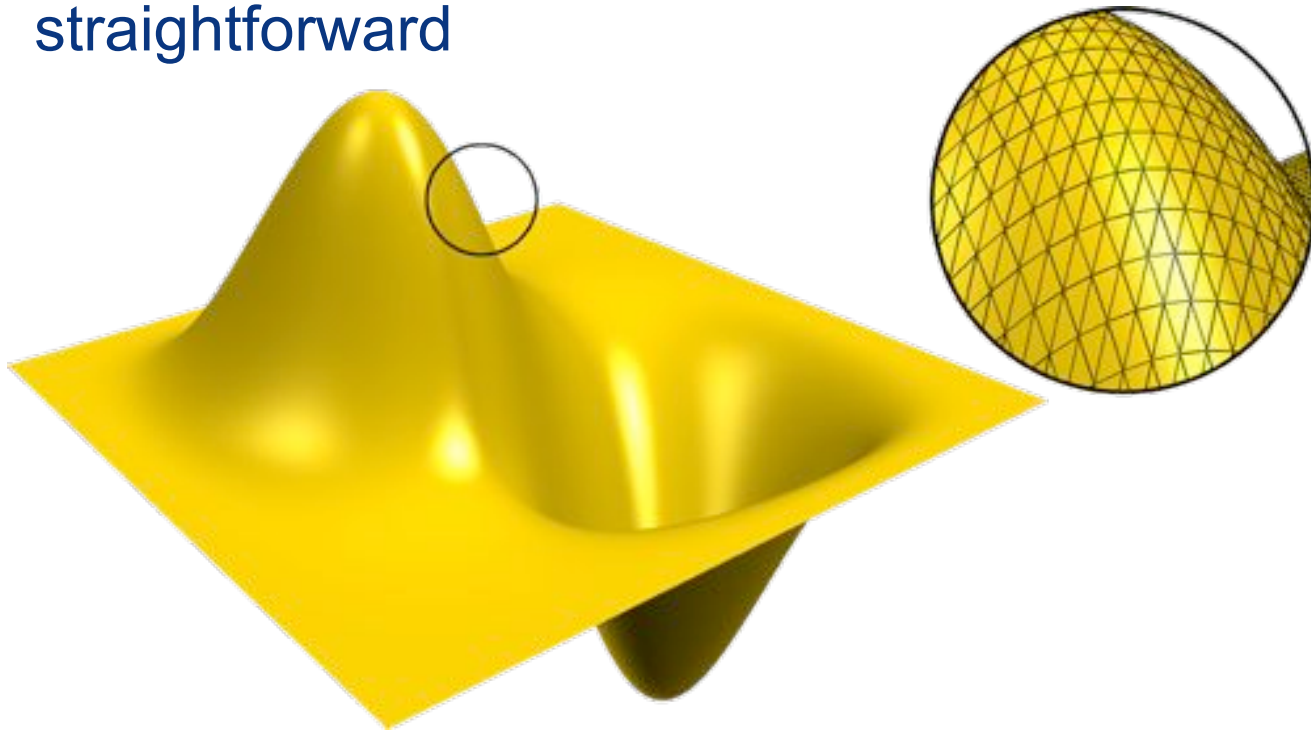
Obtain different boundary conditions



# Motivation

Higher-order equations on mesh (i.e. piecewise linear elements)

- Dealing with higher-order derivatives not straightforward



# Previous work

Simple domains, analytic boundaries

[Bloor and Wilson 1990]

Model shaped minimization of curvature variation energy

[Moreton and Séquin 1992]

Interpolate curve networks, local quadratic fits and finite differences

[Welch and Witkin 1994]

Uniform-weight discrete Laplacian

[Taubin 1995]

Cotangent-weight discrete Laplacian

[Pinkall and Polthier 1993],  
[Wardetzky et al. 2007],  
[Reuter et al. 2009]

Wilmore flow, using FEM with aux variables

- Position and co-normal specification on boundary

[Clarenz et al. 2004]

Linear systems for k-harmonic equations

- Uses discretized Laplacian operator

[Botsch and Kobbelt 04]

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# Standard Finite Element Method

Requires at least  $C^1$  elements for fourth order

- Can't use standard triangle meshes

High order surfaces exist, (e.g. Argyris triangle)

- Require **many** extra degrees of freedom
- Not popular due to complexity

Low order,  $C^0$ , workarounds

- E.g. mixed elements
-

# Discrete Geometric Discretization

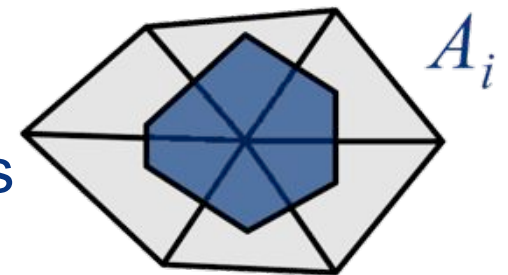
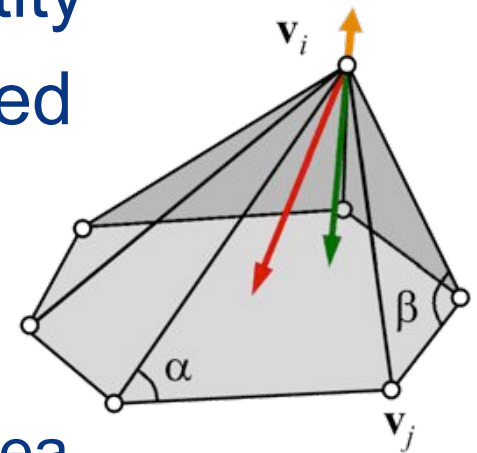
Derive mesh analog of geometric quantity

E.g. Laplace-Beltrami operator integrated over vertex area

- Re-expressed using only first-order derivatives
- Use average value as energy of vertex area

Used often in geometric modeling

- No obvious way to connect to continuous case



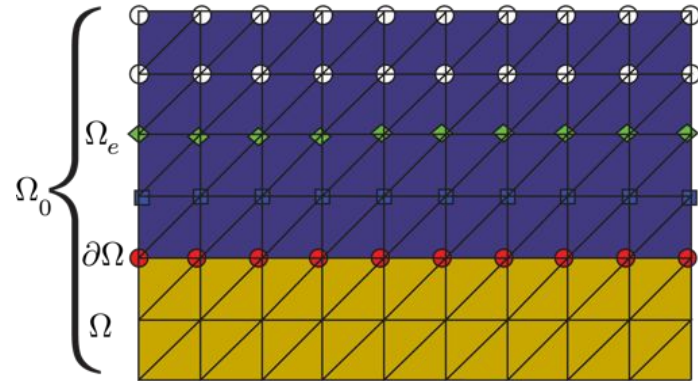


# Mixed Elements

Introduce additional variable to convert high order problem to low order

$$\frac{1}{2} \langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min \quad \rightarrow \quad \frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \Delta \mathbf{u} = \mathbf{v}$$

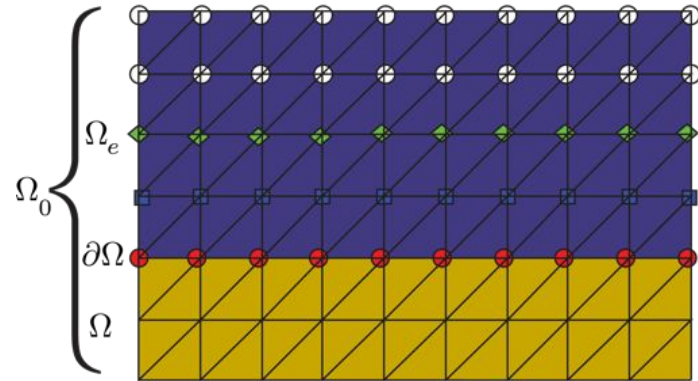
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# Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Introduce additional variable to convert high order problem to low order



$$\frac{1}{2} \langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min \rightarrow \frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \Delta \mathbf{u} = \mathbf{v}$$

Use Lagrange multipliers to enforce constraint

$$L_B = \frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \Delta \mathbf{u} - \mathbf{v} \rangle_{\Omega_0} =$$
$$\frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n} \rangle_{\partial \Omega_0} - \langle \nabla \lambda, \nabla \mathbf{u} \rangle_{\Omega_0}$$

Constraint structure also makes certain boundary types easier

# Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Our original higher order problem

$$\Delta^2 \mathbf{u} = 0$$

Introduce an additional variable

$$\Delta \mathbf{u} = \mathbf{v}$$

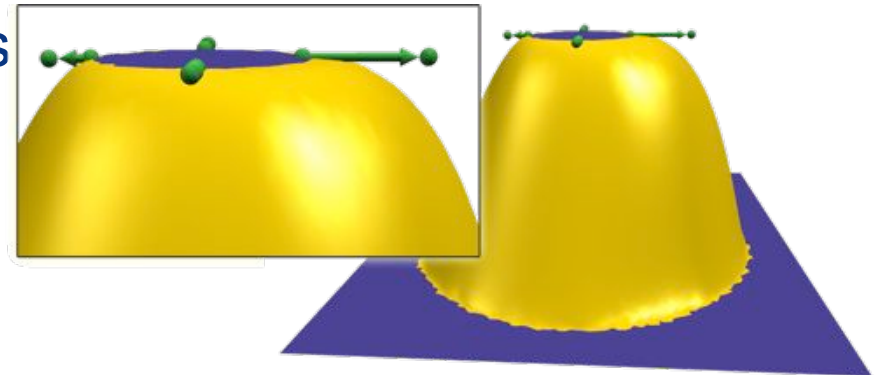
$$\Delta \mathbf{v} = 0$$

Two second order problems

- Can use just linear elements

Curve

- Fixed boundary curve  $\frac{\partial \mathbf{u}}{\partial n}$
- Specified tangents:  $\frac{\partial \mathbf{u}}{\partial n}$



# Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Discretize with piecewise linear approximations for variables

$$\sum_{j \in I} (\mathbf{v}_j - \lambda_j) \langle \phi_j, \phi_i \rangle_{\Omega_0} = 0$$

$$- \sum_{j \in I} \lambda_j \langle \nabla \phi_j, \nabla \phi_j \rangle_{\Omega_0} = 0$$

$$\sum_{j \in I} \mathbf{v}_j \langle \phi_j, \phi_i \rangle_{\Omega_0} + \sum_{j \in I} \frac{\partial \mathbf{u}_j}{\partial n} \langle \phi_j, \phi_i \rangle_{\partial \Omega_0} - \sum_{j \in I} \mathbf{u}_j \langle \nabla \phi_j, \nabla \phi_i \rangle_{\Omega_0} = 0$$

Discrete Laplacian

$$L_{ij} = -\langle \nabla \phi_i, \nabla \phi_j \rangle$$

Mass matrix

$$M_{ij}^{\text{full}} = -\langle \phi_i, \phi_j \rangle$$

# Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Discretize with piecewise linear approximations for variables

$$\sum_{j \in I} (\mathbf{v}_j - \lambda_j) \langle \phi_j, \phi_i \rangle_{\Omega_0} = 0$$

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$$\sum_{j \in I} \mathbf{v}_j \langle \phi_j, \phi_i \rangle_{\Omega_0} + \sum_{j \in I} \frac{\partial \mathbf{u}_j}{\partial n} \langle \phi_j, \phi_i \rangle_{\partial \Omega_0} - \sum_{j \in I} \mathbf{u}_j \langle \nabla \phi_j, \nabla \phi_i \rangle_{\Omega_0} = 0$$

Discrete Laplacian

$$L_{ij} = -\langle \nabla \phi_i, \nabla \phi_j \rangle$$

Mass matrix

$$M_{ij}^{\text{full}} \approx M_{ij}^{\text{d}}$$

# Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Matrix form, curve boundary conditions

$$\begin{bmatrix} -M^\Omega & L_{\bar{\Omega},\Omega} \\ L_{\Omega,\bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_\Omega \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},0}^\Omega \mathbf{b}_0 - N_{\bar{\Omega},0}^{\partial\Omega} \mathbf{b}_1 \\ 0 \end{bmatrix}$$

Discrete Laplacian

$$L_{ij} = -\langle \nabla \phi_i, \nabla \phi_j \rangle$$

Mass matrix

$$M_{ij}^d \approx -\langle \phi_j, \phi_i \rangle$$

Neumann matrix

$$N_{ij}^{\partial\Omega} = \langle \phi_j, \phi_i \rangle_{\partial\Omega}$$

Where  $\mathbf{u} = \mathbf{b}_0$  and  $\frac{\partial \mathbf{u}}{\partial n} = \mathbf{b}_n$

Diagonalized, lumped mass matrices eliminate auxiliary variable

$$L_{\Omega,\bar{\Omega}} (M^d)^{-1} L_{\bar{\Omega},\Omega} \mathbf{u}_\Omega = -L_{\Omega,\bar{\Omega}} (M^d)^{-1} (-L_{\bar{\Omega},0}^\Omega \mathbf{b}_0 - N_{\bar{\Omega},0}^{\partial\Omega} \mathbf{b}_1)$$

# Boundary Conditions

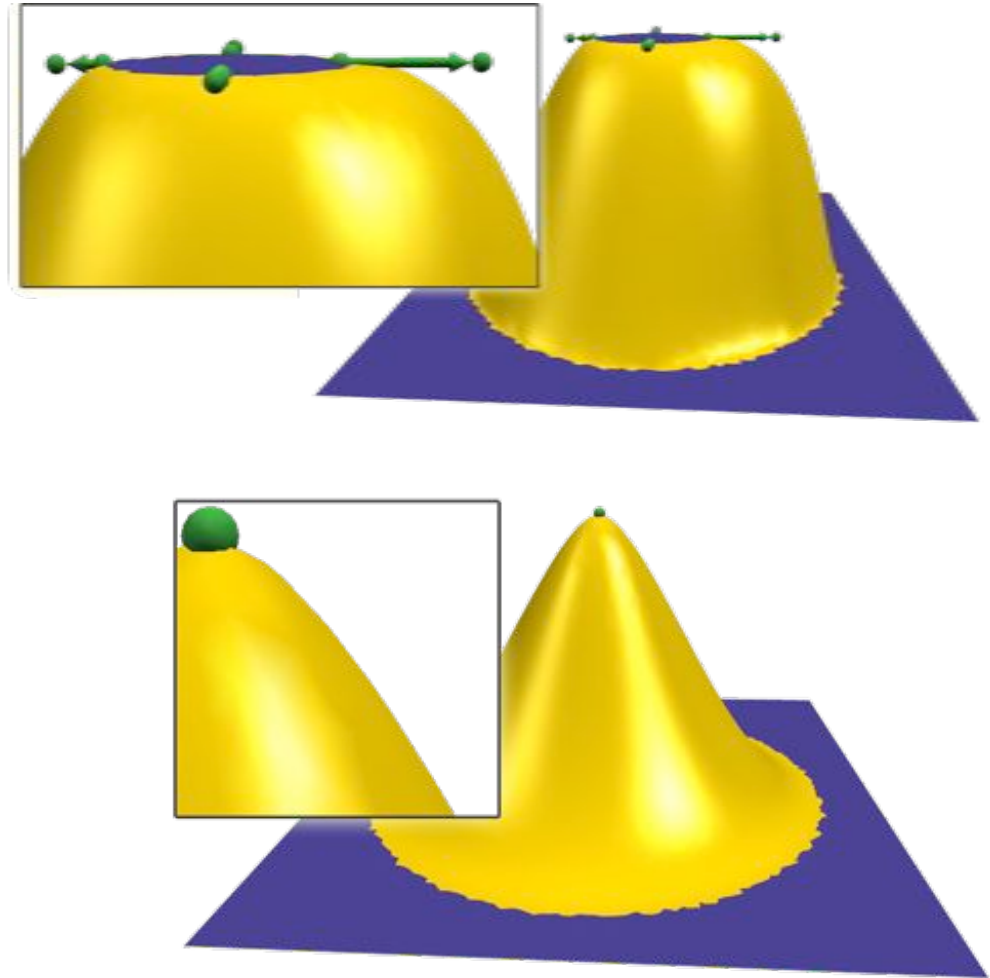
$$\Delta^2 \mathbf{u} = 0$$

## Curve

- Fixed boundary curve
- Specified tangents:  $\frac{\partial \mathbf{u}}{\partial n}$

## Point

- Single fixed points on surface

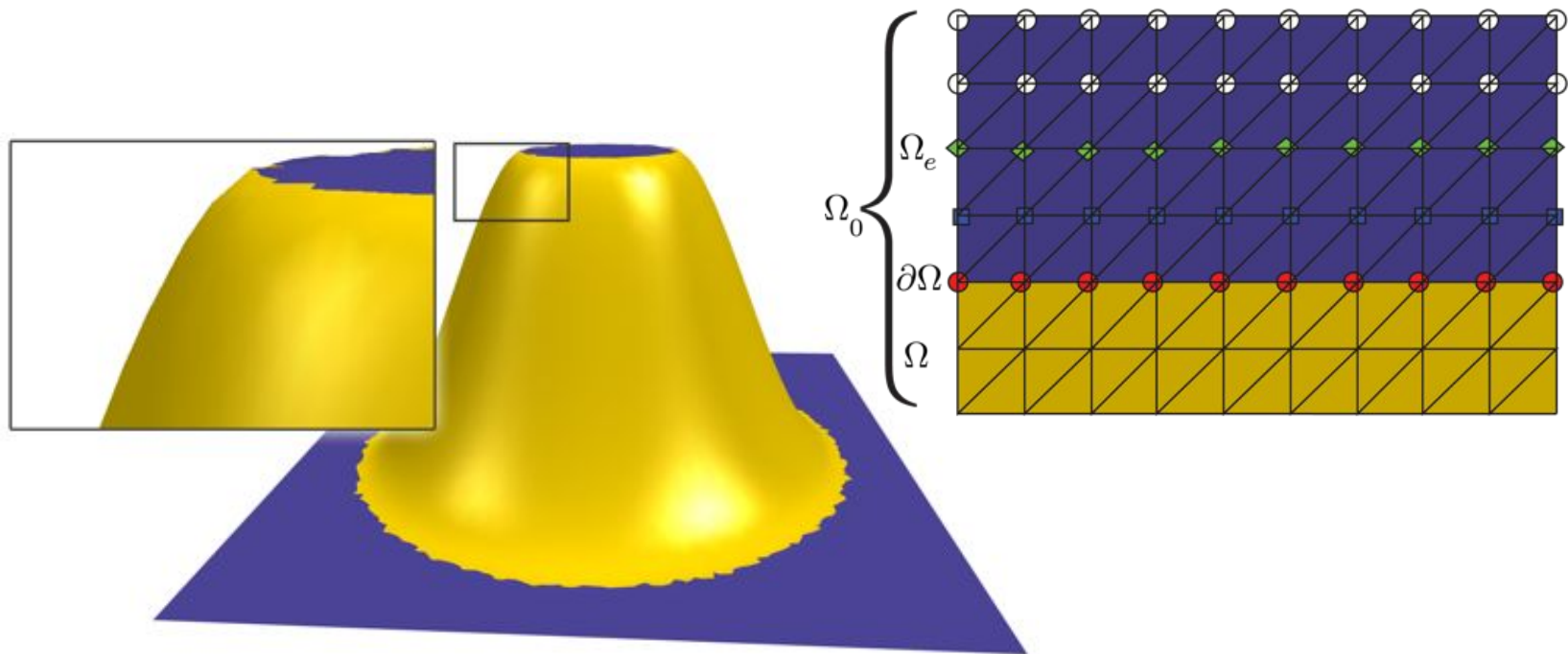


# Boundary Conditions

$$\Delta^2 \mathbf{u} = 0$$

## Region

- Fixed part of mesh outside solved region





# Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Use Lagrangian to enforce region condition

$$L_B = \frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} - \langle \lambda, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n} \rangle_{\partial \Omega} - \langle \nabla \lambda, \nabla \mathbf{u} \rangle_{\Omega_0} + \langle \mu, \mathbf{u}_f - \mathbf{u} \rangle_{\Omega_f}$$

Discretize with piecewise linear approximations for variables

$$\begin{bmatrix} -M^d & L_{\bar{\Omega}, \Omega} \\ L_{\Omega, \bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega}, 0} \mathbf{u}_0^f - L_{\bar{\Omega}, 1} \mathbf{u}_1^f \\ 0 \end{bmatrix}$$

Discrete Laplacian

$$L_{ij} = -\langle \nabla \phi_i, \nabla \phi_j \rangle$$

Mass matrix

$$M_{ij}^d \approx -\langle \phi_j, \phi_i \rangle$$

May also eliminate aux. variable

$$L_{\Omega, \bar{\Omega}} (M^d)^{-1} L_{\bar{\Omega}, \Omega} \mathbf{u}_{\Omega} = -L_{\Omega, \bar{\Omega}} (M^d)^{-1} L_{\bar{\Omega}, 01} \mathbf{u}^f$$

# Boundary Conditions

$$\Delta^2 \mathbf{u} = 0$$

Region:

$$\begin{bmatrix} -M^d & L_{\bar{\Omega},\Omega} \\ L_{\Omega,\bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},0} \mathbf{u}_0^f - L_{\bar{\Omega},1} \mathbf{u}_1^f \\ 0 \end{bmatrix}$$

Curve:

$$\begin{bmatrix} -M^{\Omega} & L_{\bar{\Omega},\Omega} \\ L_{\Omega,\bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},0}^{\Omega} \mathbf{b}_0 - N_{\bar{\Omega},0}^{\partial\Omega} \mathbf{b}_1 \\ 0 \end{bmatrix}$$

Difference in right-hand side

Curve conditions don't require lumped mass matrix

- But we use it in practice, for speed and numerical accuracy

Equivalent to [Botsch and Kobbelt, 2004]

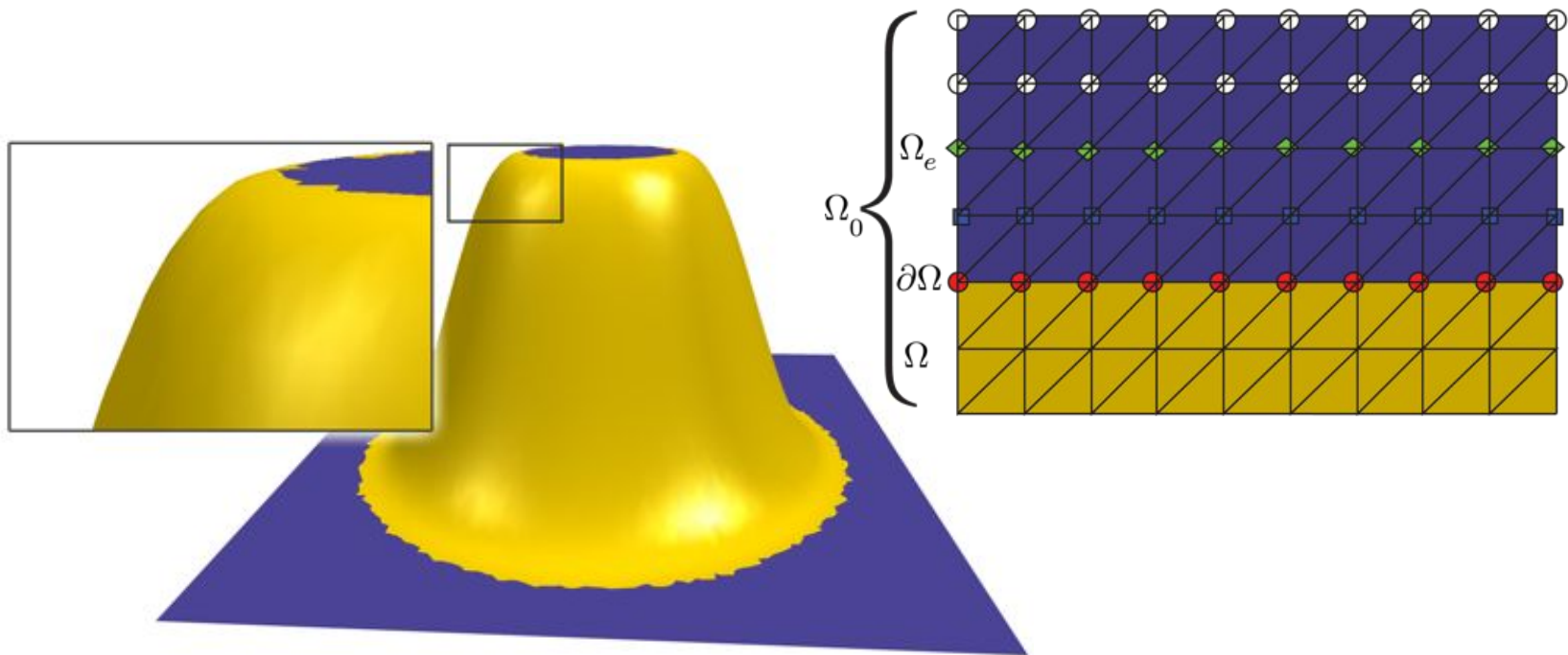
- Specified tangents  $\approx$  parameter for continuity control

# Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

## Region

- Fixed part of mesh outside solved region



# Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

Convert high order problem to low order problem

$$\frac{1}{2} \langle \nabla \Delta \mathbf{u}, \nabla \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min \rightarrow \frac{1}{2} \langle \nabla \mathbf{v}, \nabla \mathbf{v} \rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \Delta \mathbf{u} = \mathbf{v}$$

Use Langrange multipliers to enforce constraint

$$\begin{aligned} L_T &= \frac{1}{2} \langle \nabla \mathbf{v}, \nabla \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \Delta \mathbf{u} - \mathbf{v} \rangle_{\Omega_0} = \\ &\frac{1}{2} \langle \nabla \mathbf{v}, \nabla \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n} \rangle_{\partial \Omega_0} - \langle \nabla \lambda, \nabla \mathbf{u} \rangle_{\Omega_0} \end{aligned}$$

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# Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

Convert high order problem to low order problem

$$\frac{1}{2} \langle \nabla \Delta \mathbf{u}, \nabla \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min \rightarrow \frac{1}{2} \langle \nabla \mathbf{v}, \nabla \mathbf{v} \rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \Delta \mathbf{u} = \mathbf{v}$$

Use Lagrange multipliers to enforce constraint

$$L_T = \frac{1}{2} \langle \nabla \mathbf{v}, \nabla \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \Delta \mathbf{u} - \mathbf{v} \rangle_{\Omega_0} =$$
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Notice similarity to Lagrangian for biharmonic

$$L_B = \frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \Delta \mathbf{u} - \mathbf{v} \rangle_{\Omega_0} =$$
$$\frac{1}{2} \langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n} \rangle_{\partial \Omega_0} - \langle \nabla \lambda, \nabla \mathbf{u} \rangle_{\Omega_0}$$

# Mixed Elements

$$\Delta^3 \mathbf{u} = 0$$

Discretization, formulation works the same way

$$\begin{bmatrix} L_{\bar{\Omega}\bar{\Omega}} & 0 & -M_{\bar{\Omega}\bar{\Omega}}^d \\ 0 & 0 & L_{\Omega\bar{\Omega}} \\ -M_{\bar{\Omega}\bar{\Omega}}^d & L_{\bar{\Omega}\Omega} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \\ \lambda_{\bar{\Omega}} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},1} \mathbf{v}_1 \\ 0 \\ -L_{\bar{\Omega},0} \mathbf{u}_0^f - L_{\bar{\Omega},1} \mathbf{u}_1^f \end{bmatrix}$$

where  $\mathbf{v}_1 = -L_{1,0} \mathbf{u}_0^f - L_{1,1} \mathbf{u}_1^f - L_{1,2} \mathbf{u}_2^f$

Eliminate auxiliary variables

- Leaving system with only  $\mathbf{u}$

Discrete Laplacian

$$L_{ij} = -\langle \nabla \phi_i, \nabla \phi_j \rangle$$

Mass matrix

$$M_{ij}^{\text{full}} = -\langle \phi_i, \phi_j \rangle$$

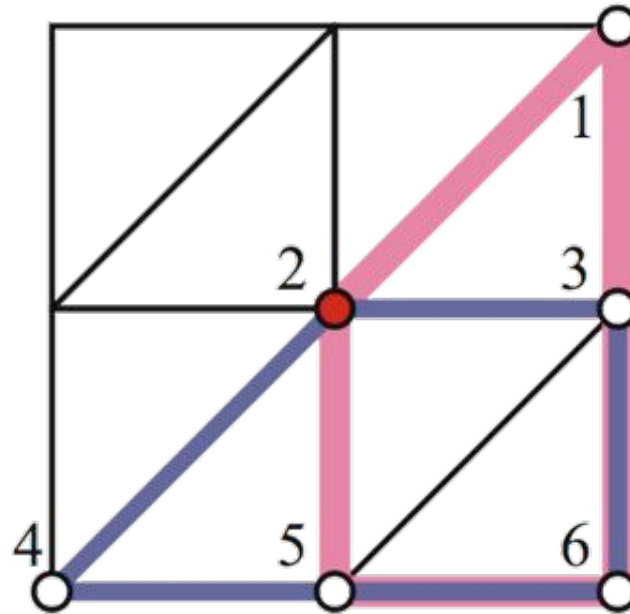
# Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

## Curve

- Fixed boundary curve
- Specified tangents and curvatures:  $\frac{\partial \mathbf{u}}{\partial n}$ ,  $\frac{\partial^2 \mathbf{u}}{\partial n^2}$

Leads to singular systems



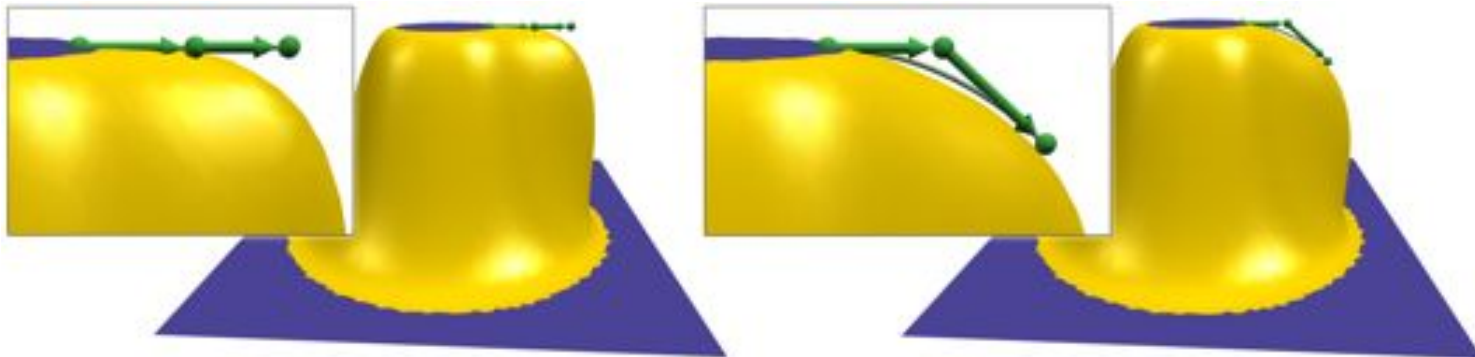
# Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

Curve  $\rightarrow$  Region

- Fixed boundary curve and one ring into interior
- Specified curvatures:  $\frac{\partial^2 \mathbf{u}}{\partial n^2}$

$$\mathbf{v} = \Delta \mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial n^2} + \frac{\partial^2 \mathbf{u}}{\partial t^2}$$



$$\begin{bmatrix} L_{\bar{\Omega}\bar{\Omega}} & 0 & -M_{\bar{\Omega}\bar{\Omega}}^d \\ 0 & 0 & L_{\Omega\bar{\Omega}} \\ -M_{\bar{\Omega}\bar{\Omega}}^d & L_{\bar{\Omega}\Omega} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \\ \lambda_{\bar{\Omega}} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},1} \mathbf{v}_1 \\ 0 \\ -L_{\bar{\Omega},0} \mathbf{u}_0^f - L_{\bar{\Omega},1} \mathbf{u}_1^f \end{bmatrix}$$

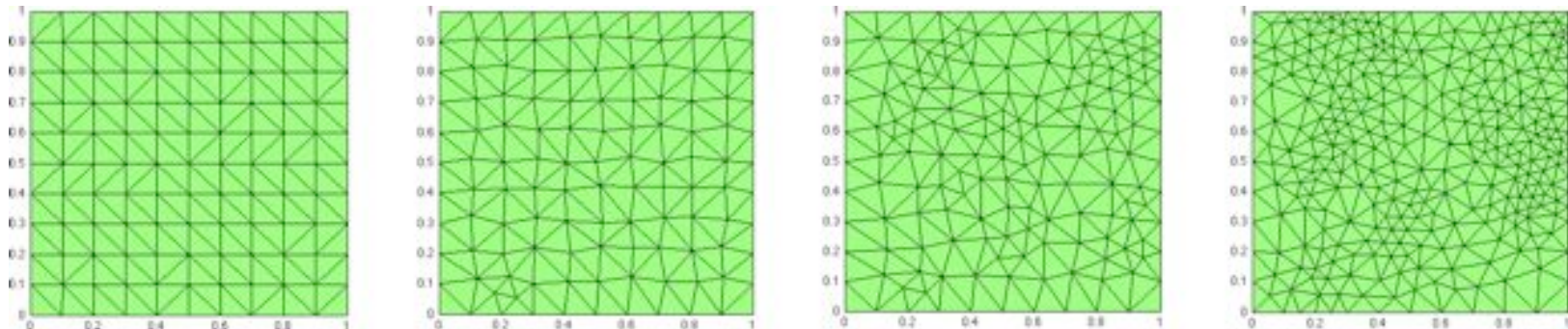


# Experimental Results

Tested convergence of our systems

Randomly generated domains of varying irregularity

- One vertex placed randomly in each square of grid
- Parameter controlled variation from regular



Connected using Triangle Library

- Control minimal interior angles
-

# Experimental Results

Specify boundary conditions using analytic target functions:  $\mathbf{u}^t$

- Try to reproduce original function by solving system:

$$\Delta^2 \mathbf{u} = \Delta^2 \mathbf{u}^t$$

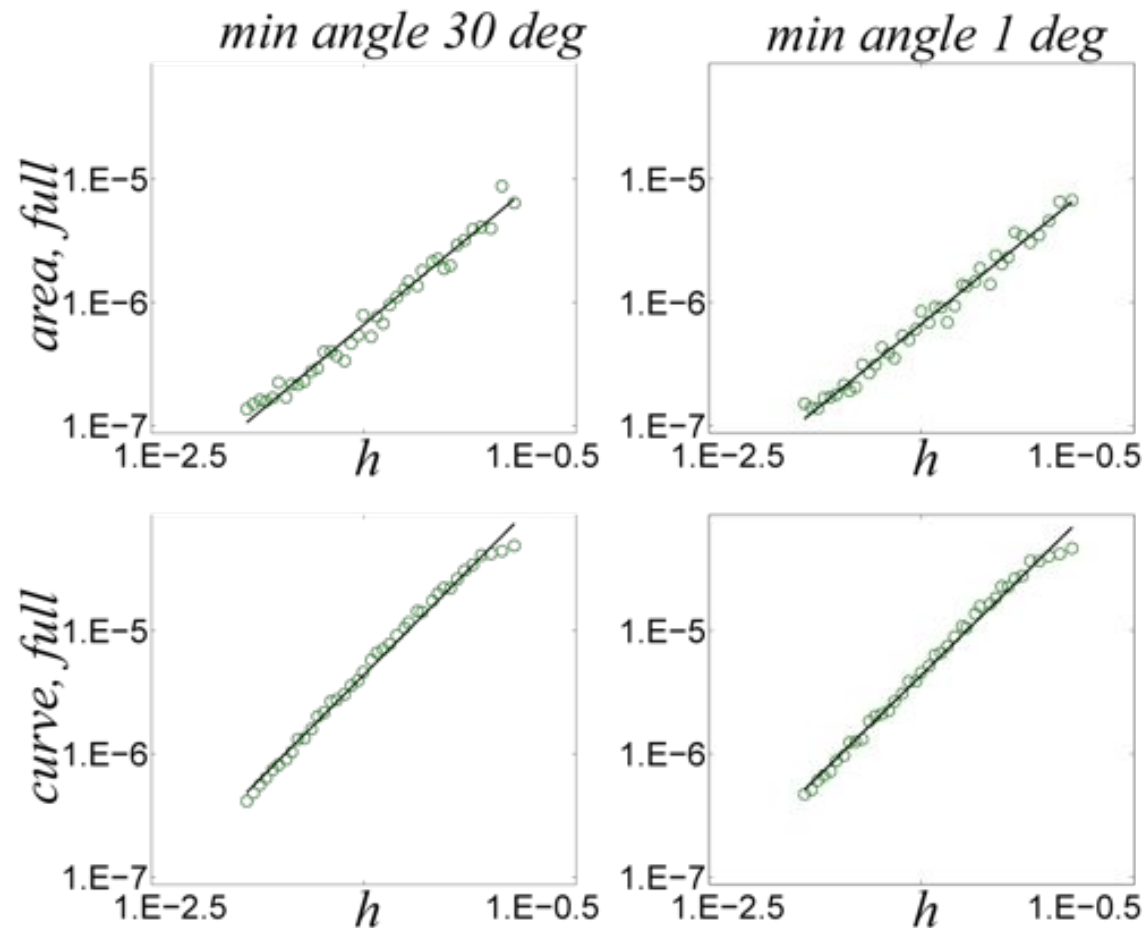
Measure error between analytic target and our mixed FEM approximation

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# Experimental Results

$$\Delta^2 \mathbf{u} = 0$$

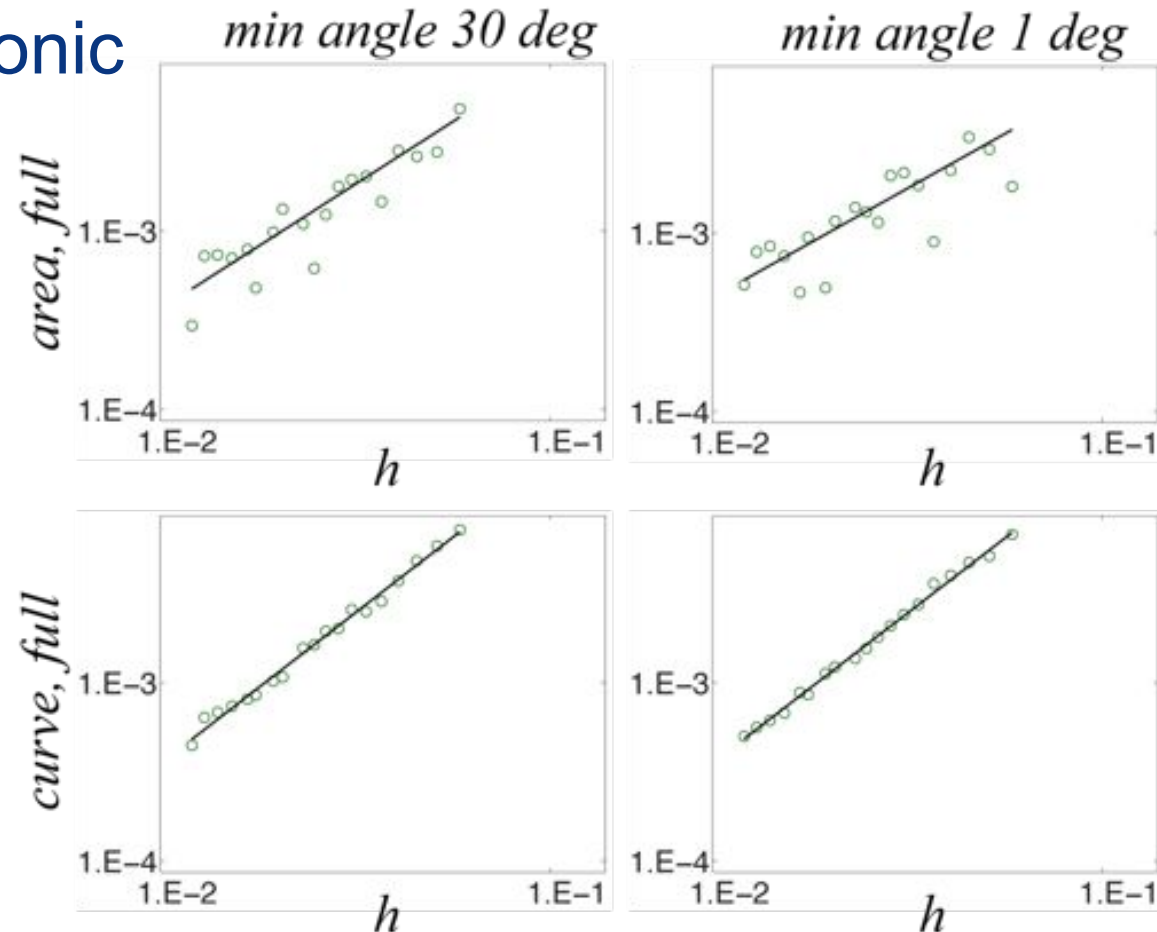
Nearly optimal convergence for biharmonic



# Experimental Results

$$\Delta^3 \mathbf{u} = 0$$

Boundary conditions perform differently for triharmonic



# Applications

$$\Delta^2 \mathbf{u} = 0$$

Filling in holes: Laplacian energy

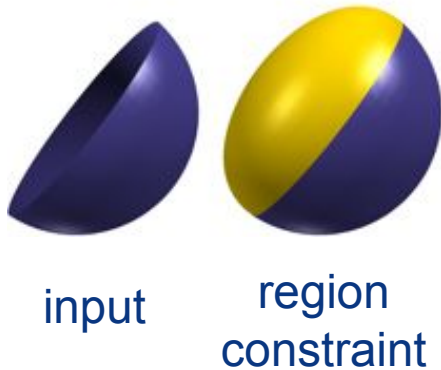


input

# Applications

$$\Delta^2 \mathbf{u} = 0$$

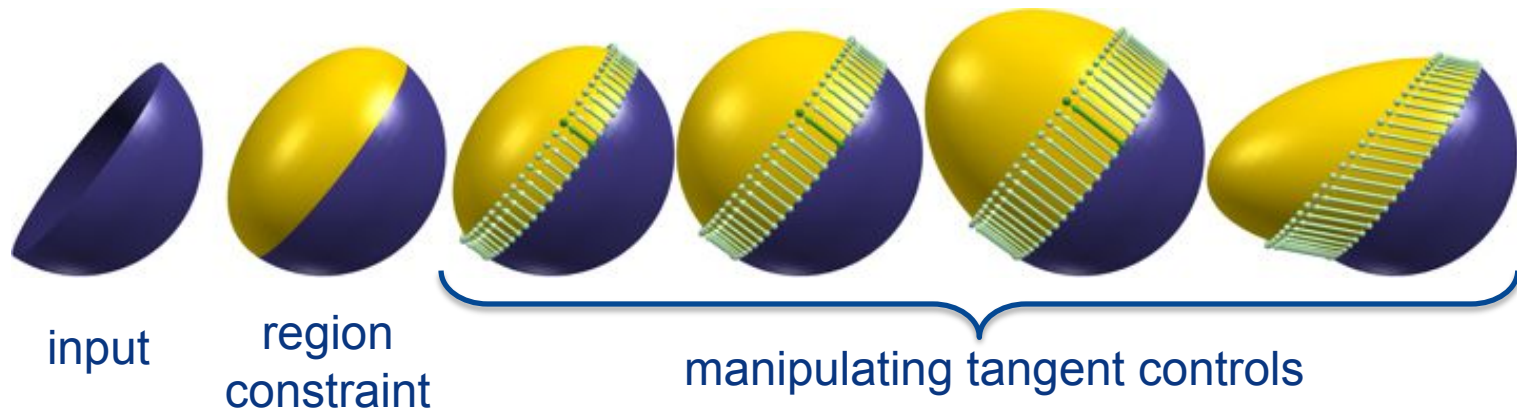
Filling in holes: Laplacian energy



# Applications

$$\Delta^2 \mathbf{u} = 0$$

Filling in holes: Laplacian energy



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Filling in holes: Laplacian energy



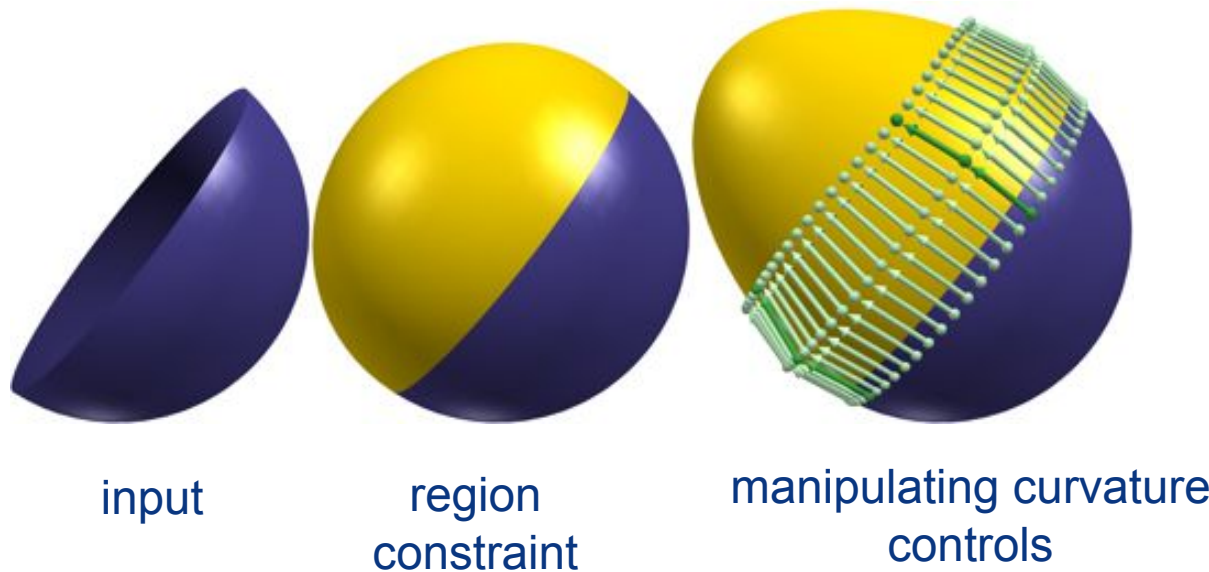
manipulating tangent controls



# Applications

$$\Delta^3 \mathbf{u} = 0$$

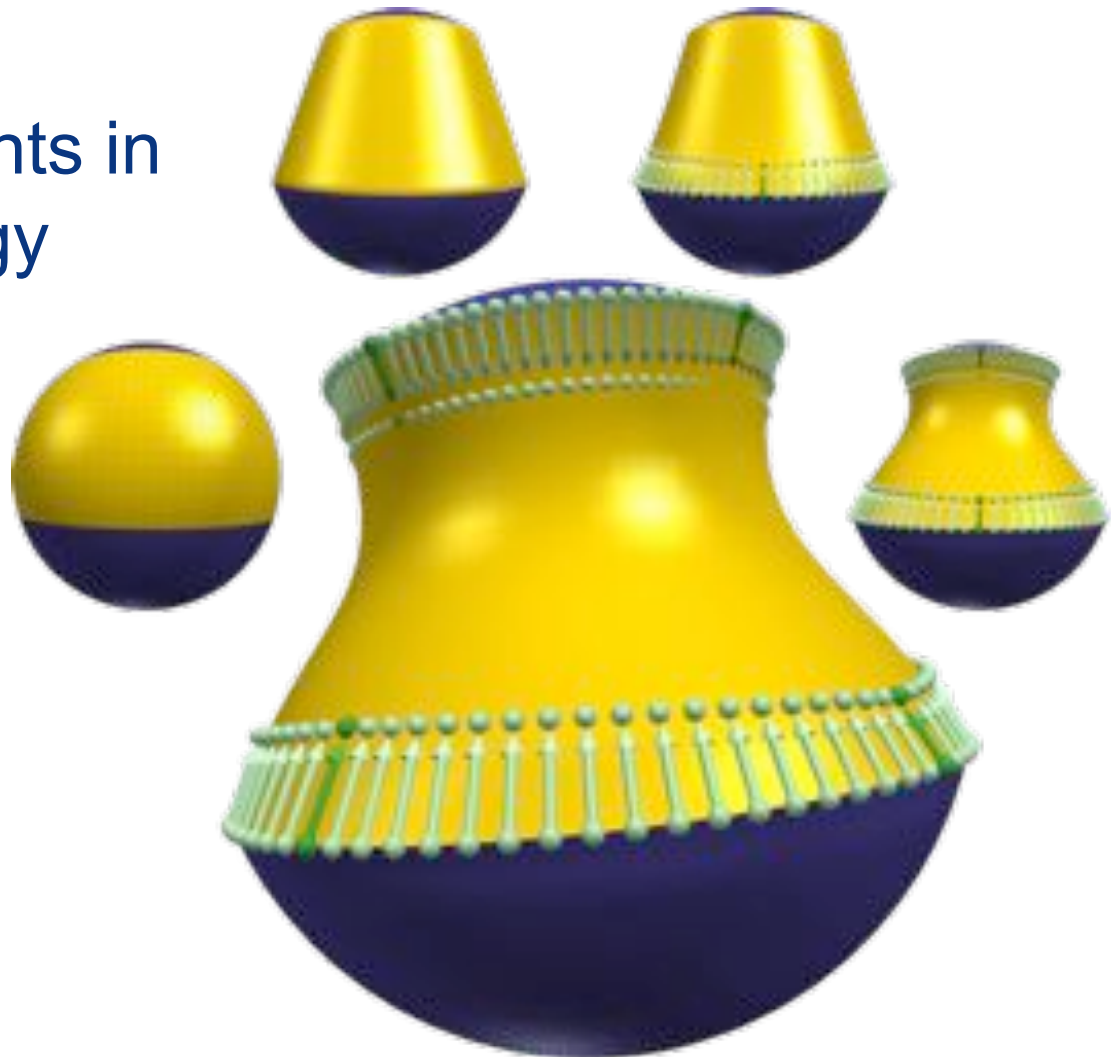
Filling in holes: Laplacian gradient energy



# Applications

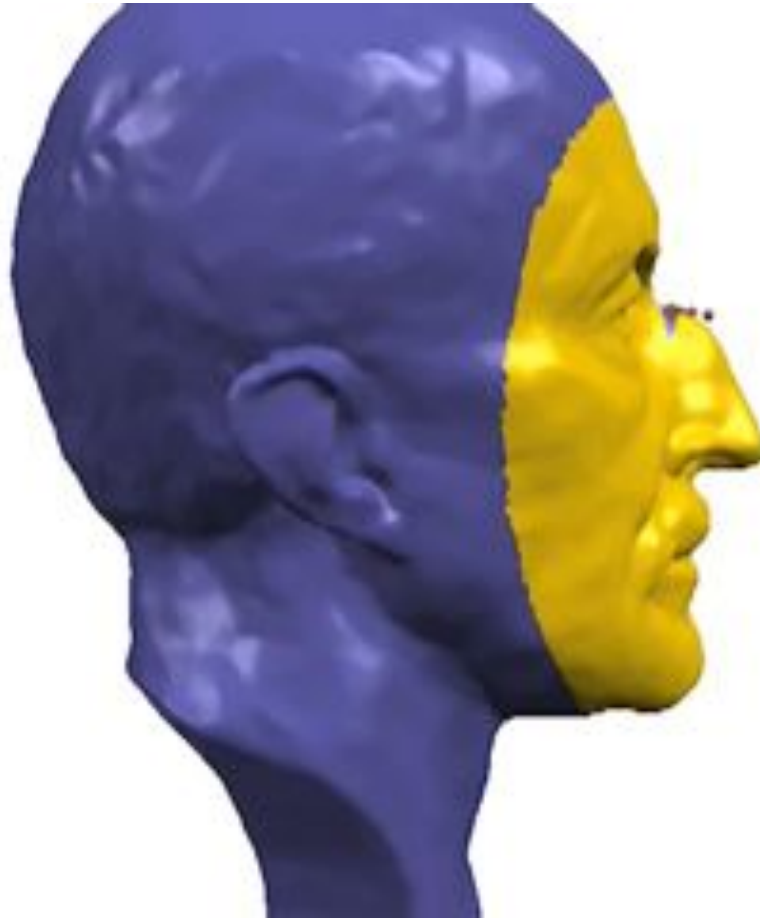
$$\Delta^2 \mathbf{u} = 0$$

Specifying tangents in  
Laplacian energy  
around regions



# Applications

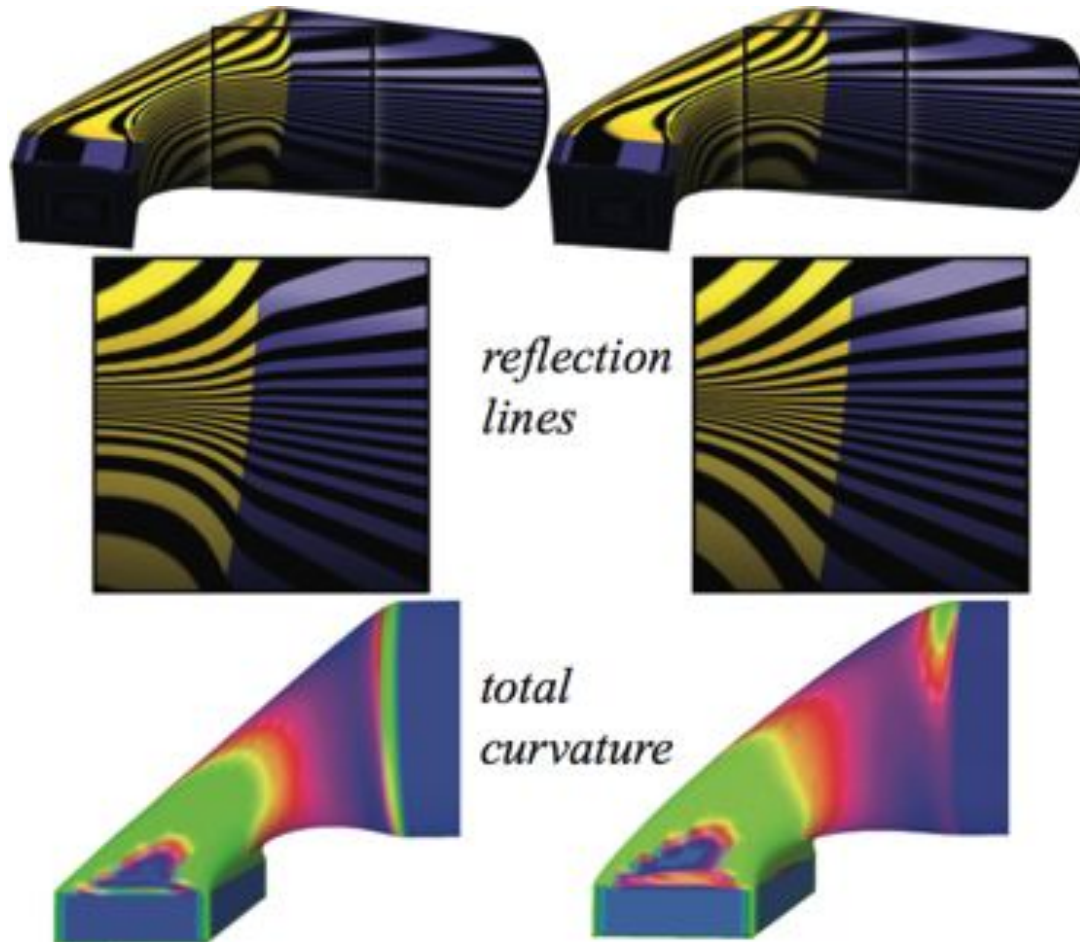
$$\Delta^3 \mathbf{u} = 0$$



# Applications

Biharmonic

Triharmonic





# Summary

## Technique for discretizing energies or PDEs

- Reduce to low order by introducing variables
- Use constraints to enforce region boundary conditions
- Lump mass matrix

## Convergence for fourth- and sixth-order PDEs

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# Summary

Technique for discretizing energies or PDEs

- Reduce to low order by introducing variables
- Use constraints to enforce region boundary conditions
- Lump mass matrix

Convergence for fourth- and sixth-order PDEs

Future work

- Improve convergence of triharmonic solution
  - Explore using non-flat metric
-



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# Mixed Finite Elements for Variational Surface Modeling

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