Mixed Finite Elements for Variational Surface Modeling

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Motivation

$$\langle f, g \rangle = \int fg$$

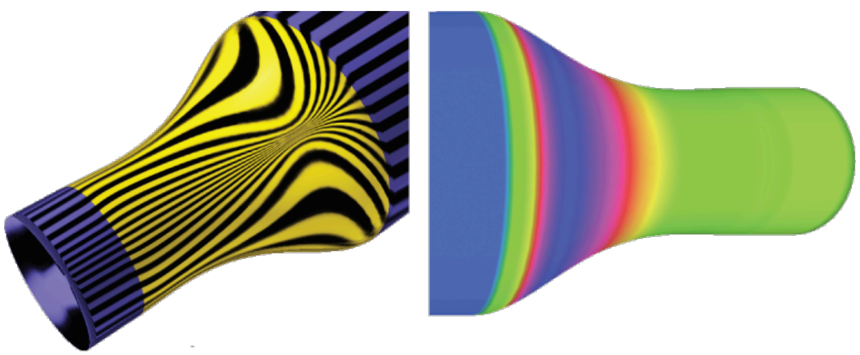
Produce high-quality surfaces

Via energy minimization

Or solving Partial Differential Equations

Laplacian energy

$$E_B = \frac{1}{2}\langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min$$

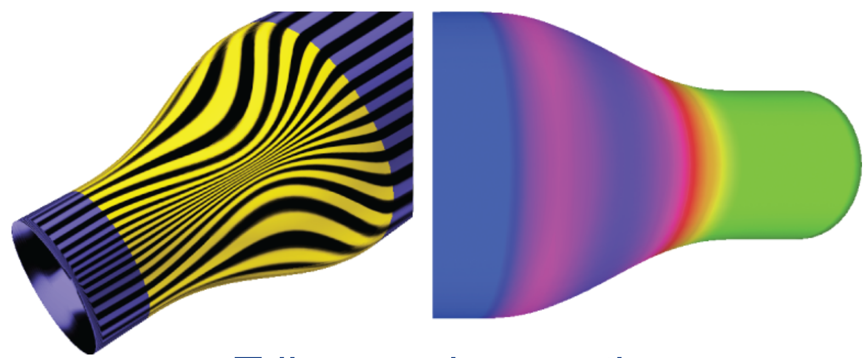


Biharmonic equation

$$\Delta^2 \mathbf{u} = 0$$

Laplacian gradient energy

$$E_T = \frac{1}{2}\langle \nabla\Delta \mathbf{u}, \nabla\Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min$$

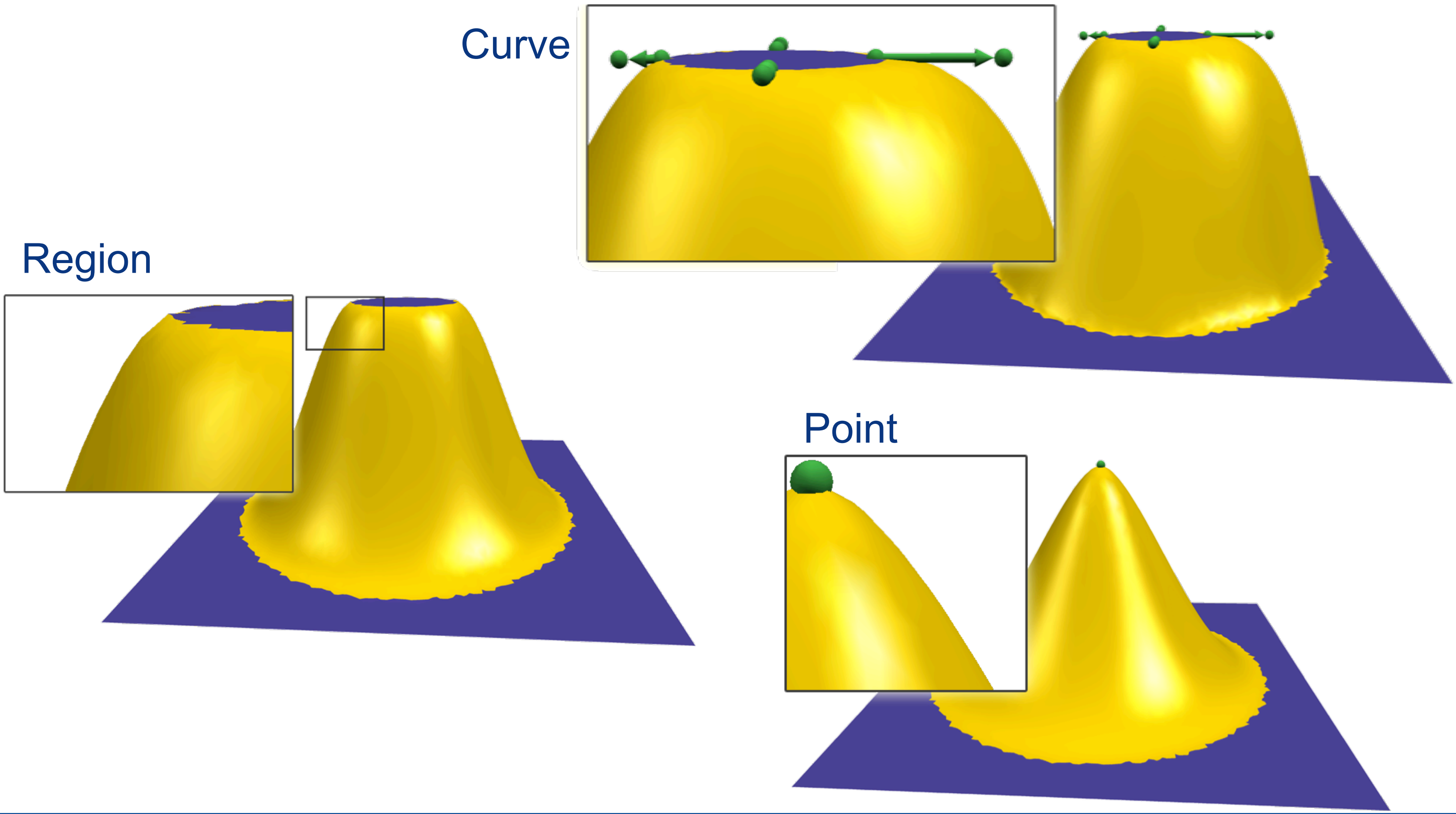


Triharmonic equation

$$\Delta^3 \mathbf{u} = 0$$

Motivation

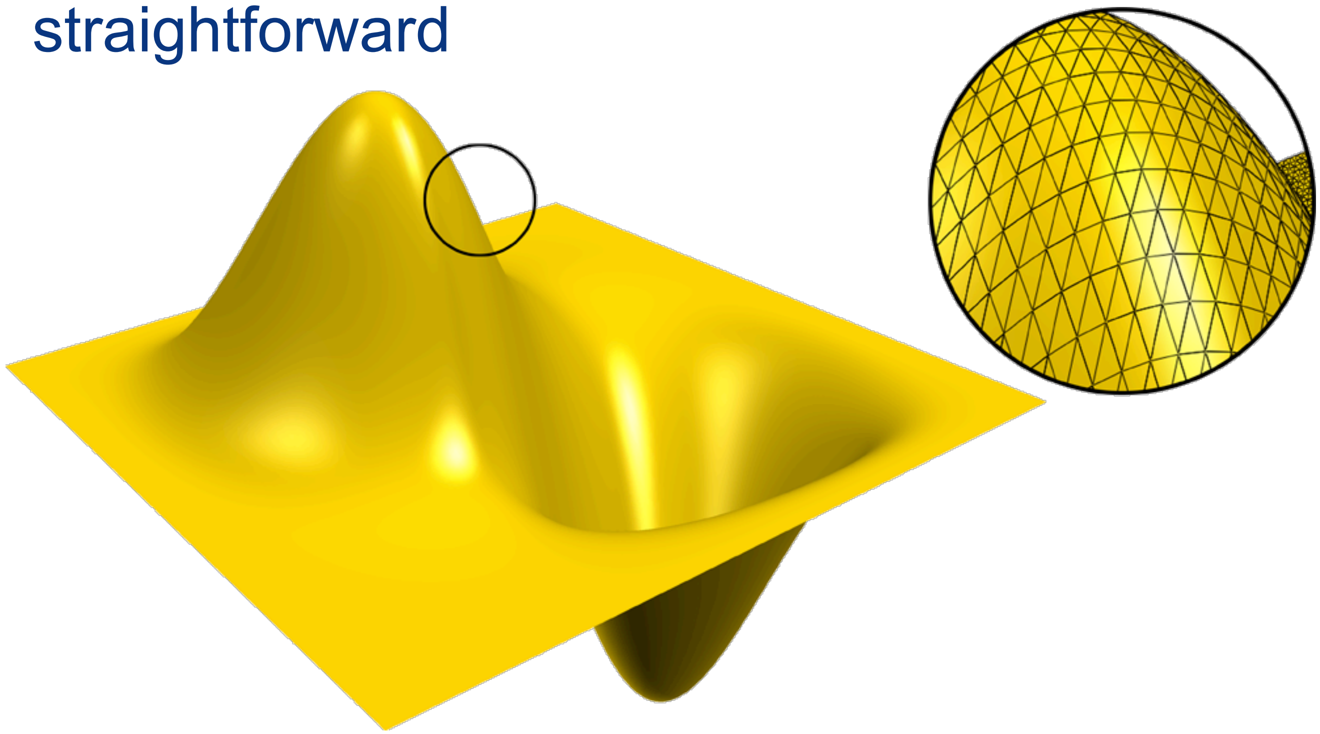
Obtain different boundary conditions



Motivation

Higher-order equations on mesh (i.e. piecewise linear elements)

- Dealing with higher-order derivatives not straightforward



Previous work

Simple domains, analytic boundaries

[Bloor and Wilson 1990]

Model shaped minimization of curvature variation energy

[Moreton and Séquin 1992]

Interpolate curve networks, local quadratic fits and finite differences

[Welch and Witkin 1994]

Uniform-weight discrete Laplacian

[Taubin 1995]

Cotangent-weight discrete Laplacian

[Pinkall and Polthier 1993],
[Wardetzky et al. 2007],
[Reuter et al. 2009]

Wilmore flow, using FEM with aux variables

- Position and co-normal specification on boundary

[Clarenz et al. 2004]

Linear systems for k-harmonic equations

- Uses discretized Laplacian operator

[Botsch and Kobbelt 04]

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Standard Finite Element Method

Requires at least C^1 elements for fourth order

- Can’t use standard triangle meshes

High order surfaces exist, (e.g. Argyris triangle)

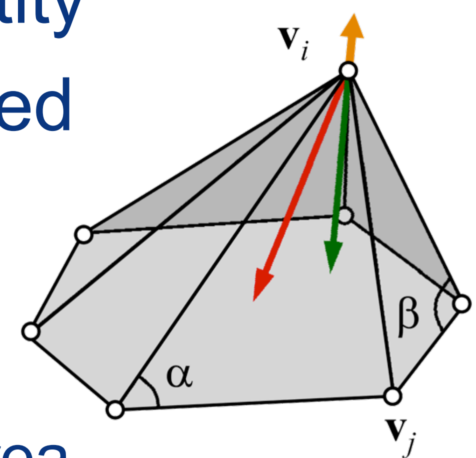
- Require **many** extra degrees of freedom
- Not popular due to complexity

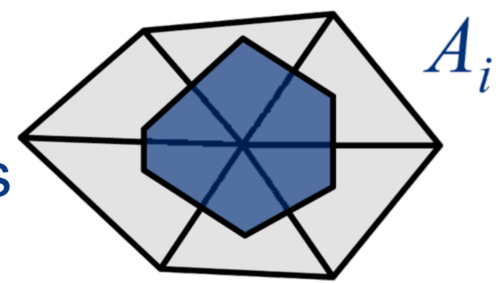
Low order, C^0, workarounds

- E.g. mixed elements

Discrete Geometric Discretization

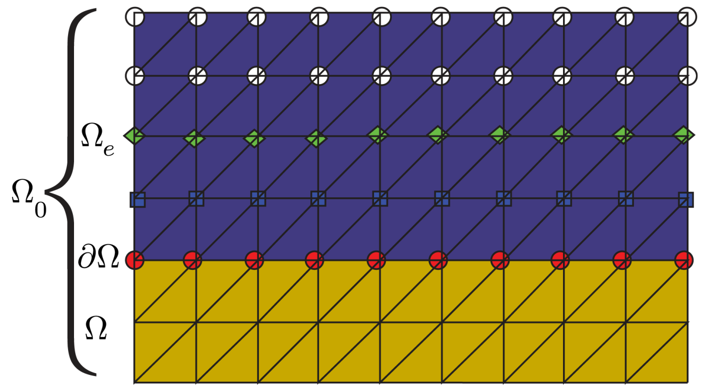
- Derive mesh analog of geometric quantity
- E.g. Laplace-Beltrami operator integrated over vertex area
  - Re-expressed using only first-order derivatives
  - Use average value as energy of vertex area
- Used often in geometric modeling
  - No obvious way to connect to continuous case





Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

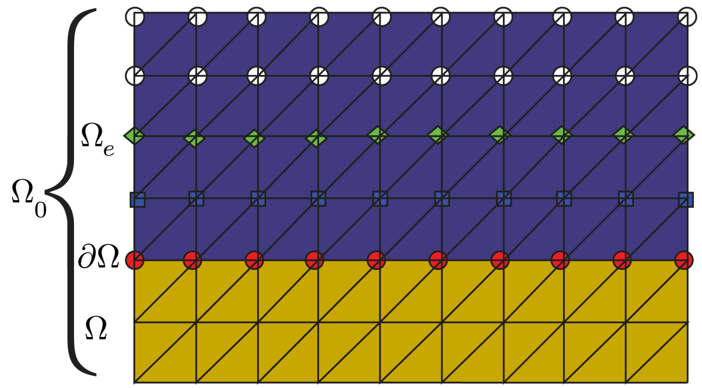


Introduce additional variable to convert high order problem to low order

$$\frac{1}{2}\langle \Delta \mathbf{u}, \Delta \mathbf{u} \rangle_{\Omega_0} \rightarrow \min \quad \rightarrow \quad \frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \Delta \mathbf{u} = \mathbf{v}$$

Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$



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Use Langrange multipliers to enforce constraint

$$L_B = \frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \Delta\mathbf{u} - \mathbf{v}\rangle_{\Omega_0} =$$

$$\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n}\rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u}\rangle_{\Omega_0}$$

Constraint structure also makes certain boundary types easier

Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Our original higher order problem

$$\Delta^2 \mathbf{u} = 0$$

Introduce an additional variable

$$\Delta \mathbf{u} = \mathbf{v}$$

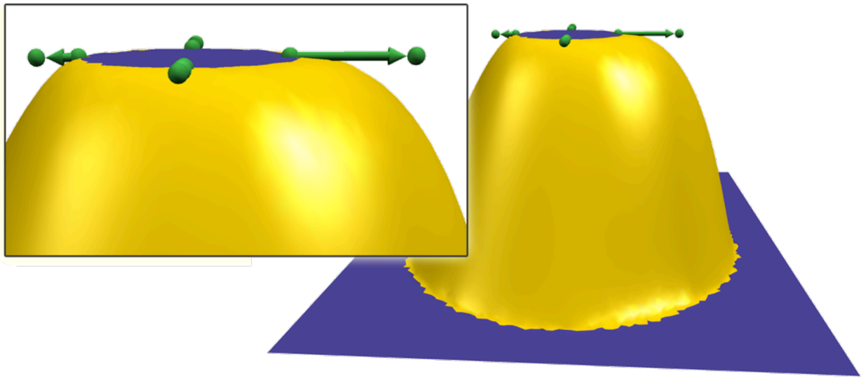
$$\Delta \mathbf{v} = 0$$

Two second order problems

- Can use just linear elements

Curve

- Fixed boundary curve
- Specified tangents: $\dfrac{\partial \mathbf{u}}{\partial n}$



Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Discretize with piecewise linear approximations for variables

$$\sum_{j \in I} (\mathbf{v}_j - \lambda_j) \langle \phi_j, \phi_i \rangle_{\Omega_0} = 0$$

$$-\sum_{j \in I} \lambda_j \langle \nabla \phi_j, \nabla \phi_j \rangle_{\Omega_0} = 0$$

$$\sum_{j \in I} \mathbf{v}_j \langle \phi_j, \phi_i \rangle_{\Omega_0} + \sum_{j \in I} \frac{\partial \mathbf{u}_j}{\partial n} \langle \phi_j, \phi_i \rangle_{\partial \Omega_0} - \sum_{j \in I} \mathbf{u}_j \langle \nabla \phi_j, \nabla \phi_i \rangle_{\Omega_0} = 0$$

Discrete Laplacian

$$L_{ij} = -\langle \nabla \phi_i, \nabla \phi_j \rangle$$

Mass matrix

$$M_{ij}^{\text{full}} = -\langle \phi_i, \phi_j \rangle$$

Mixed Elements

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$$\sum_{j \in I} (\mathbf{v}_j - \lambda_j) \langle \phi_j, \phi_i \rangle_{\Omega_0} = 0$$

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Discrete Laplacian

$$L_{ij} = -\langle \nabla \phi_i, \nabla \phi_j \rangle$$

Mass matrix

$$M_{ij}^{\text{full}} \approx M_{ij}^{\text{d}}$$

Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Matrix form, curve boundary conditions

$$\begin{bmatrix} -M^{\Omega} & L_{\bar{\Omega},\Omega} \\ L_{\Omega,\bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \end{bmatrix} = \begin{bmatrix} -L^{\Omega}_{\bar{\Omega},0}\mathbf{b}_0 - N^{\partial\Omega}_{\bar{\Omega},0}\mathbf{b}_1 \\ 0 \end{bmatrix}$$

Discrete Laplacian
$L_{ij} = -\langle \nabla\phi_i, \nabla\phi_j \rangle$

Mass matrix
$M^{\mathrm{d}}_{ij} \approx -\langle \phi_j, \phi_i \rangle$

Neumann matrix
$N^{\partial\Omega}_{ij} = \langle \phi_j, \phi_i \rangle_{\partial\Omega}$

Where $\mathbf{u} = \mathbf{b}_0$ and $\dfrac{\partial \mathbf{u}}{\partial n} = \mathbf{b}_n$

Diagonalized, lumped mass matrices eliminate auxiliary variable

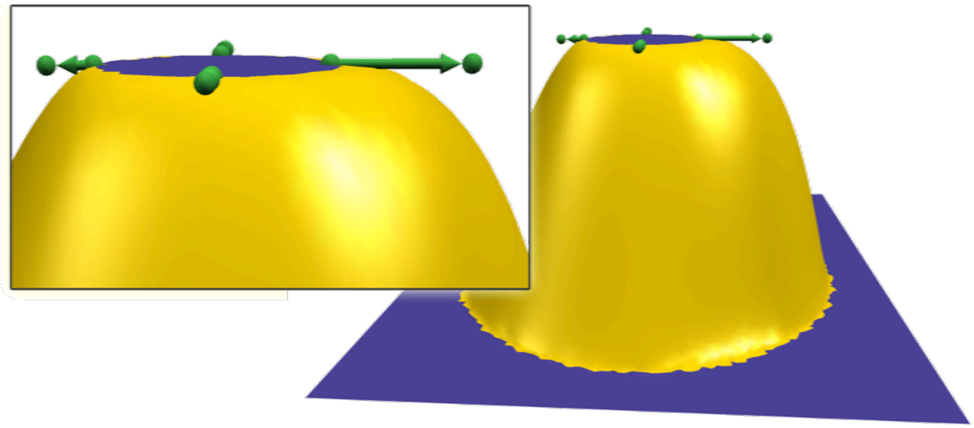
$$L_{\Omega,\bar{\Omega}}(M^{\mathrm{d}})^{-1}L_{\bar{\Omega},\Omega}\mathbf{u}_{\Omega} = -L_{\Omega,\bar{\Omega}}(M^{\mathrm{d}})^{-1}(-L^{\Omega}_{\bar{\Omega},0}\mathbf{b}_0 - N^{\partial\Omega}_{\bar{\Omega},0}\mathbf{b}_1)$$

Boundary Conditions

$$\Delta^2 \mathbf{u} = 0$$

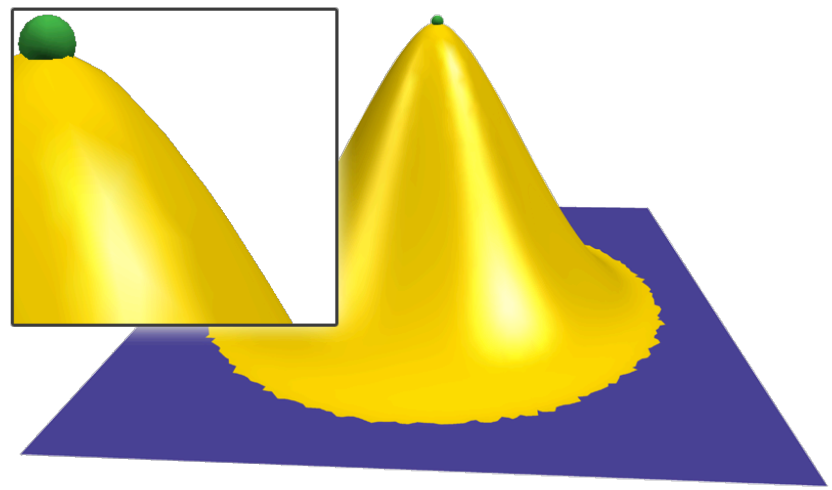
Curve

- Fixed boundary curve
- Specified tangents: $\dfrac{\partial \mathbf{u}}{\partial n}$



Point

- Single fixed points on surface

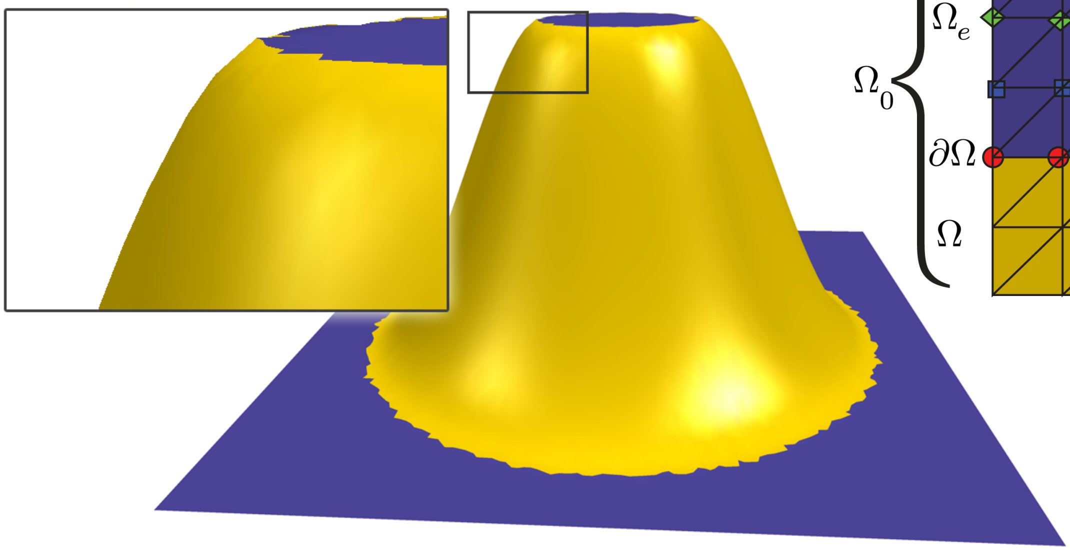


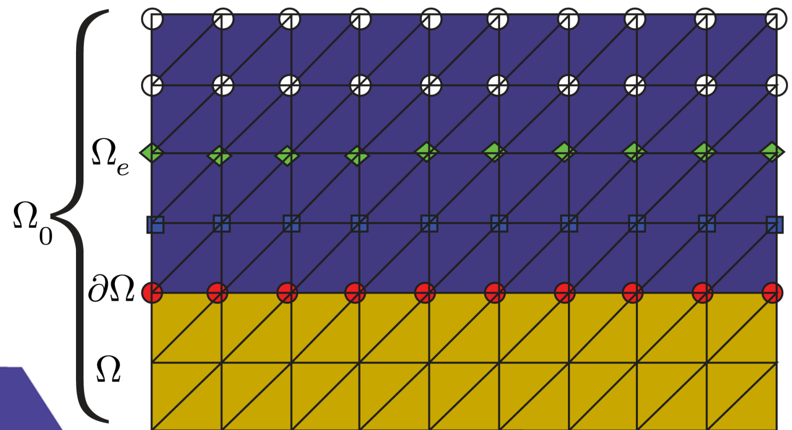
$$\Delta^2 \mathbf{u} = 0$$

Boundary Conditions

Region

- Fixed part of mesh outside solved region





Mixed Elements

$$\Delta^2 \mathbf{u} = 0$$

Use Lagrangian to enforce region condition

$$L_B = \frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle_{\Omega_0} - \langle \lambda, \mathbf{v} \rangle_{\Omega_0} + \langle \lambda, \frac{\partial \mathbf{u}}{\partial n} \rangle_{\partial\Omega} - \langle \nabla\lambda, \nabla\mathbf{u} \rangle_{\Omega_0} + \langle \mu, \mathbf{u}_f - \mathbf{u} \rangle_{\Omega_f}$$

Discretize with piecewise linear approximations for variables

$$\begin{bmatrix} -M^d & L_{\bar{\Omega},\Omega} \\ L_{\Omega,\bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},0}\mathbf{u}_0^f - L_{\bar{\Omega},1}\mathbf{u}_1^f \\ 0 \end{bmatrix}$$

Discrete Laplacian
$$L_{ij} = -\langle \nabla\phi_i, \nabla\phi_j \rangle$$

Mass matrix
$$M_{ij}^{\mathrm{d}} \approx -\langle \phi_j, \phi_i \rangle$$

May also eliminate aux. variable

$$L_{\Omega,\bar{\Omega}}(M^{\mathrm{d}})^{-1} L_{\bar{\Omega},\Omega}\mathbf{u}_{\Omega} = -L_{\Omega,\bar{\Omega}}(M^{\mathrm{d}})^{-1} L_{\bar{\Omega},01}\mathbf{u}^f$$

Boundary Conditions

$$\Delta^2 \mathbf{u} = 0$$

Region:

$$\begin{bmatrix} -M^d & L_{\bar{\Omega},\Omega} \\ L_{\Omega,\bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},0}\mathbf{u}_0^f - L_{\bar{\Omega},1}\mathbf{u}_1^f \\ 0 \end{bmatrix}$$

Curve:

$$\begin{bmatrix} -M^\Omega & L_{\bar{\Omega},\Omega} \\ L_{\Omega,\bar{\Omega}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \end{bmatrix} = \begin{bmatrix} -L^{\Omega}_{\bar{\Omega},0}\mathbf{b}_0 - N^{\partial\Omega}_{\bar{\Omega},0}\mathbf{b}_1 \\ 0 \end{bmatrix}$$

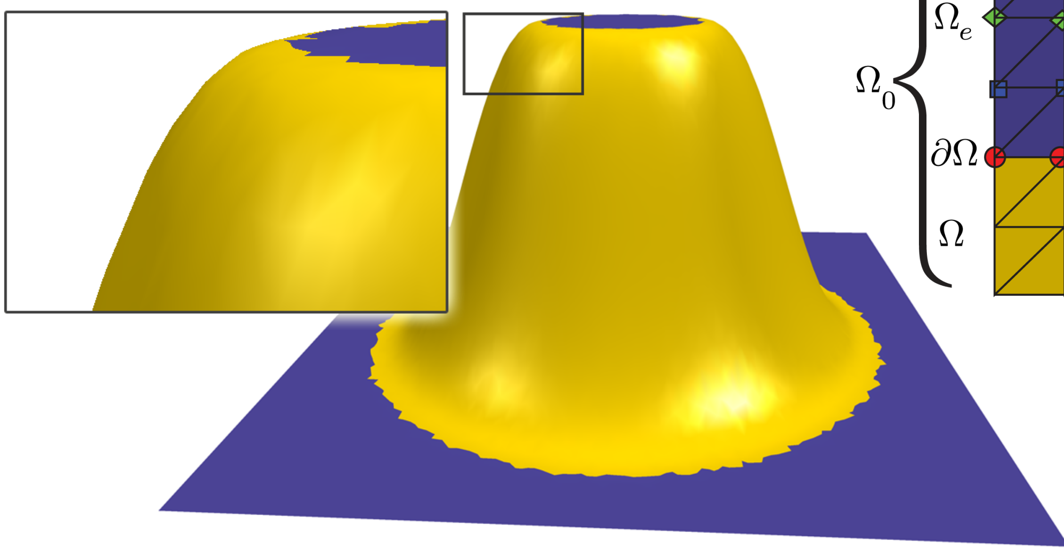
- Difference in right-hand side
- Curve conditions don’t require lumped mass matrix
  - But we use it in practice, for speed and numerical accuracy
- Equivalent to [Botsch and Kobbelt, 2004]
  - Specified tangents ≈ parameter for continuity control

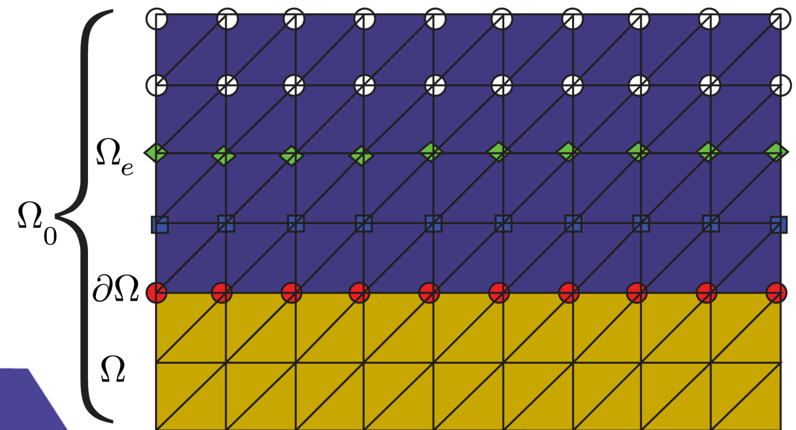
Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

Region

- Fixed part of mesh outside solved region





Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

Convert high order problem to low order problem

$$\frac{1}{2}\langle \nabla\Delta\mathbf{u}, \nabla\Delta\mathbf{u}\rangle_{\Omega_0} \rightarrow \min \quad \rightarrow \quad \frac{1}{2}\langle \nabla\mathbf{v}, \nabla\mathbf{v}\rangle_{\Omega_0} \rightarrow \min, \text{ s.t. } \Delta\mathbf{u} = \mathbf{v}$$

Use Langrange multipliers to enforce constraint

$$L_T = \frac{1}{2}\langle \nabla\mathbf{v}, \nabla\mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \Delta\mathbf{u} - \mathbf{v}\rangle_{\Omega_0} =$$

$$\frac{1}{2}\langle \nabla\mathbf{v}, \nabla\mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \frac{\partial\mathbf{u}}{\partial n}\rangle_{\partial\Omega_0} - \langle \nabla\lambda, \nabla\mathbf{u}\rangle_{\Omega_0}$$

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Notice similarity to Lagrangian for biharmonic

$$L_B = \frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle_{\Omega_0} + \langle \lambda, \Delta\mathbf{u} - \mathbf{v}\rangle_{\Omega_0} =$$

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Mixed Elements

$$\Delta^3 \mathbf{u} = 0$$

Discretization, formulation works the same way

$$\begin{bmatrix} L_{\bar{\Omega}\bar{\Omega}} & 0 & -M^d_{\bar{\Omega}\bar{\Omega}} \\ 0 & 0 & L_{\Omega\bar{\Omega}} \\ -M^d_{\bar{\Omega}\bar{\Omega}} & L_{\bar{\Omega}\Omega} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \\ \lambda_{\bar{\Omega}} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},1}\mathbf{v}_1 \\ 0 \\ -L_{\bar{\Omega},0}\mathbf{u}^f_0 - L_{\bar{\Omega},1}\mathbf{u}^f_1 \end{bmatrix}$$

where $\mathbf{v}_1 = -L_{1,0}\mathbf{u}^f_0 - L_{1,1}\mathbf{u}^f_1 - L_{1,2}\mathbf{u}^f_2$

Discrete Laplacian
$L_{ij} = -\langle \nabla\phi_i, \nabla\phi_j \rangle$

Mass matrix
$M^{\text{full}}_{ij} = -\langle \phi_i, \phi_j \rangle$

Eliminate auxiliary variables

- Leaving system with only $\mathbf{u}$

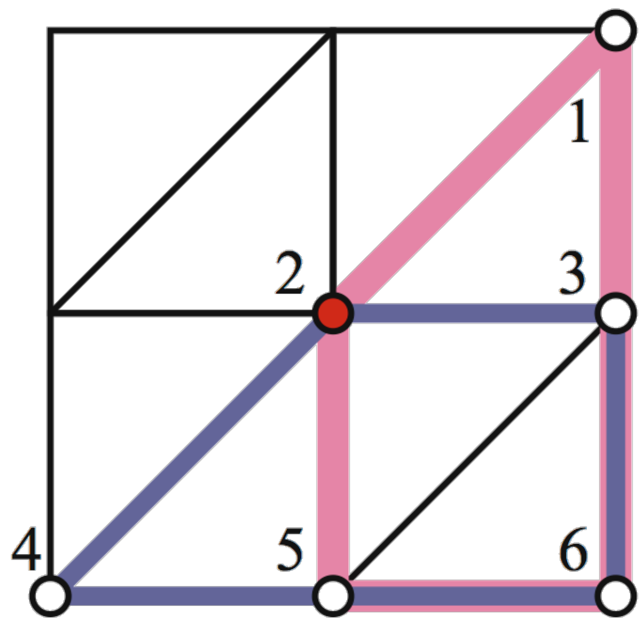
Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

Curve

- Fixed boundary curve
- Specified tangents and curvatures: $\frac{\partial \mathbf{u}}{\partial n}, \frac{\partial^2 \mathbf{u}}{\partial n^2}$

Leads to singular systems



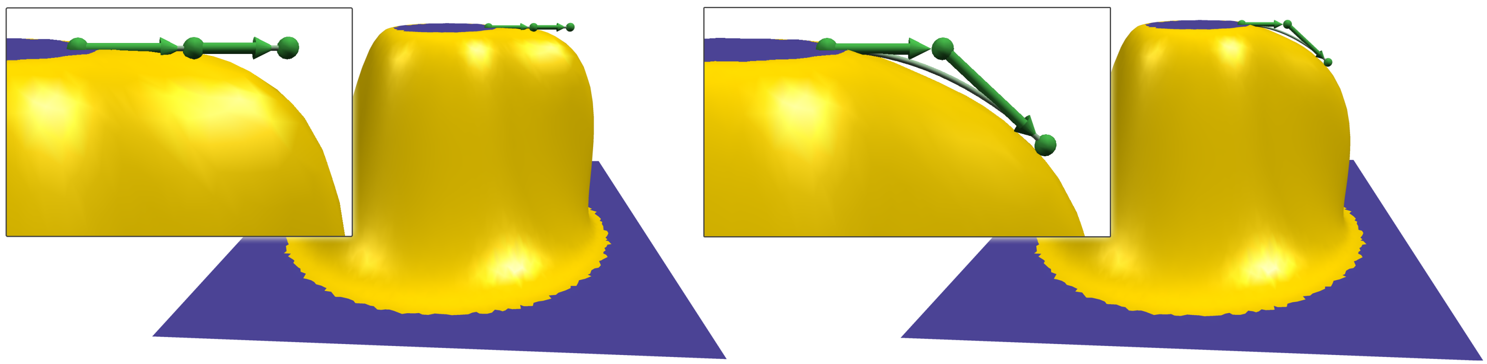
Boundary Conditions

$$\Delta^3 \mathbf{u} = 0$$

Curve → Region

- Fixed boundary curve and one ring into interior
- Specified curvatures: $\frac{\partial^2 \mathbf{u}}{\partial n^2}$

$$\mathbf{v} = \Delta \mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial n^2} + \frac{\partial^2 \mathbf{u}}{\partial t^2}$$



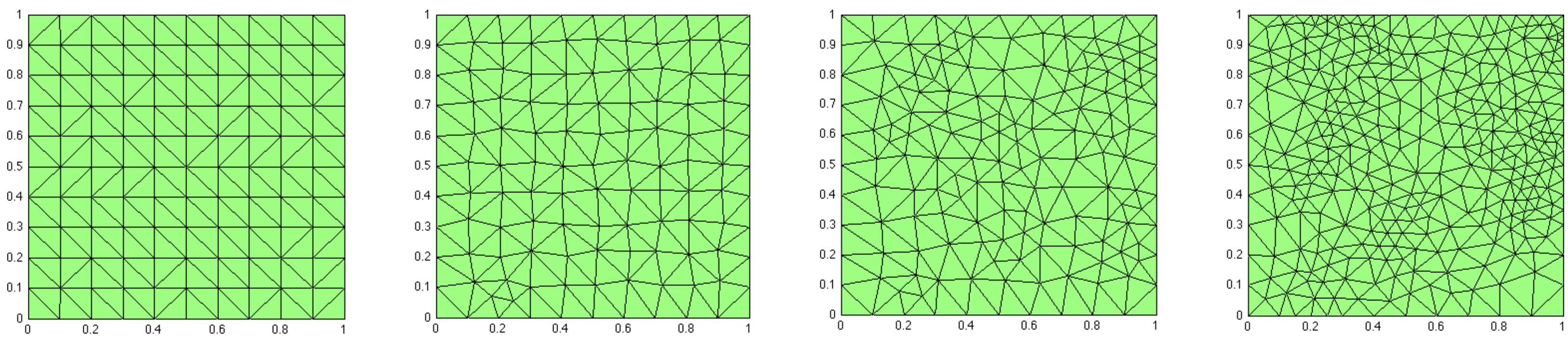
$$\begin{bmatrix} L_{\bar{\Omega}\bar{\Omega}} & 0 & -M^d_{\bar{\Omega}\bar{\Omega}} \\ 0 & 0 & L_{\Omega\bar{\Omega}} \\ -M^d_{\bar{\Omega}\bar{\Omega}} & L_{\bar{\Omega}\Omega} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\bar{\Omega}} \\ \mathbf{u}_{\Omega} \\ \lambda_{\bar{\Omega}} \end{bmatrix} = \begin{bmatrix} -L_{\bar{\Omega},1}\mathbf{v}_1 \\ 0 \\ -L_{\bar{\Omega},0}\mathbf{u}_0^f - L_{\bar{\Omega},1}\mathbf{u}_1^f \end{bmatrix}$$

Experimental Results

Tested convergence of our systems

Randomly generated domains of varying irregularity

- One vertex placed randomly in each square of grid
- Parameter controlled variation from regular



Connected using Triangle Library

- Control minimal interior angles

Experimental Results

- Specify boundary conditions using analytic target functions: $\mathbf{u}^t$
  - Try to reproduce original function by solving system:

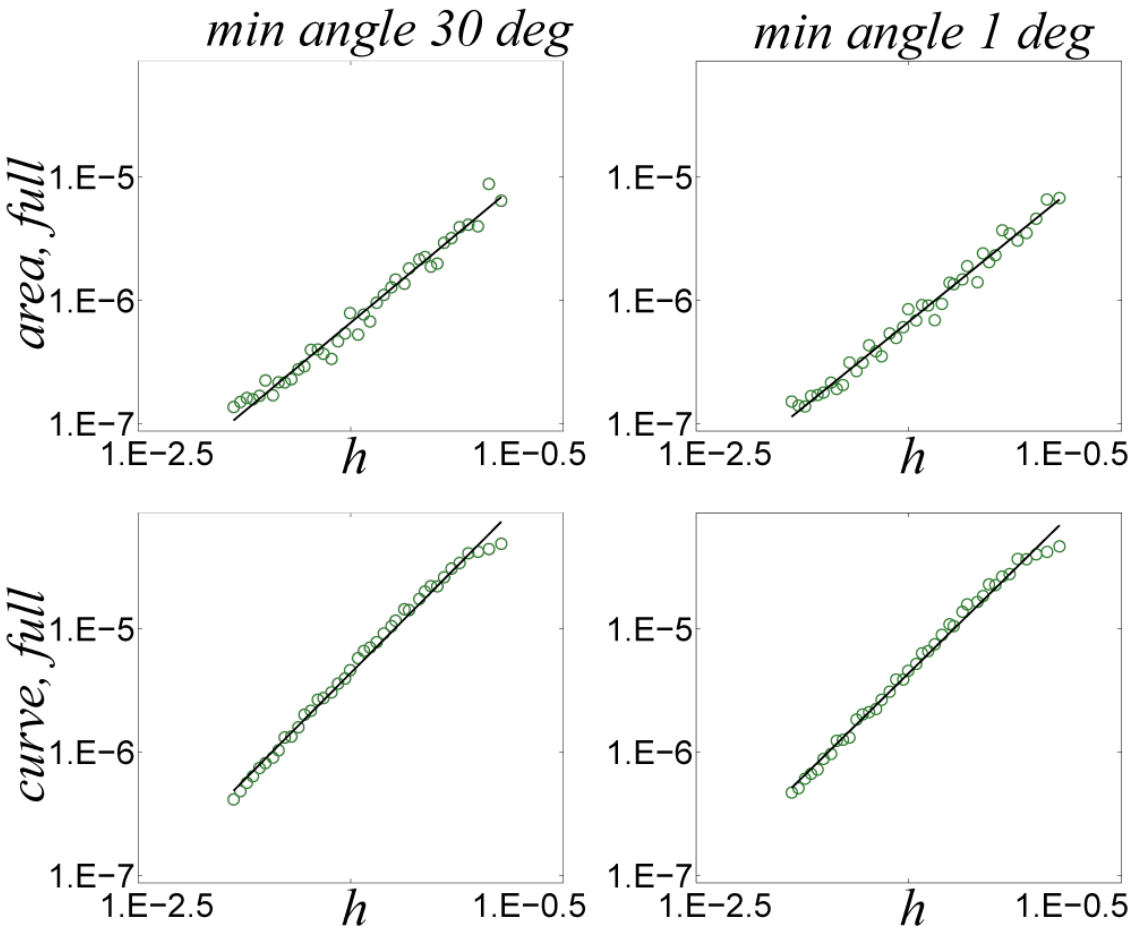
$$\Delta^2 \mathbf{u} = \Delta^2 \mathbf{u}^t$$

- Measure error between analytic target and our mixed FEM approximation

Experimental Results

$$\Delta^2 \mathbf{u} = 0$$

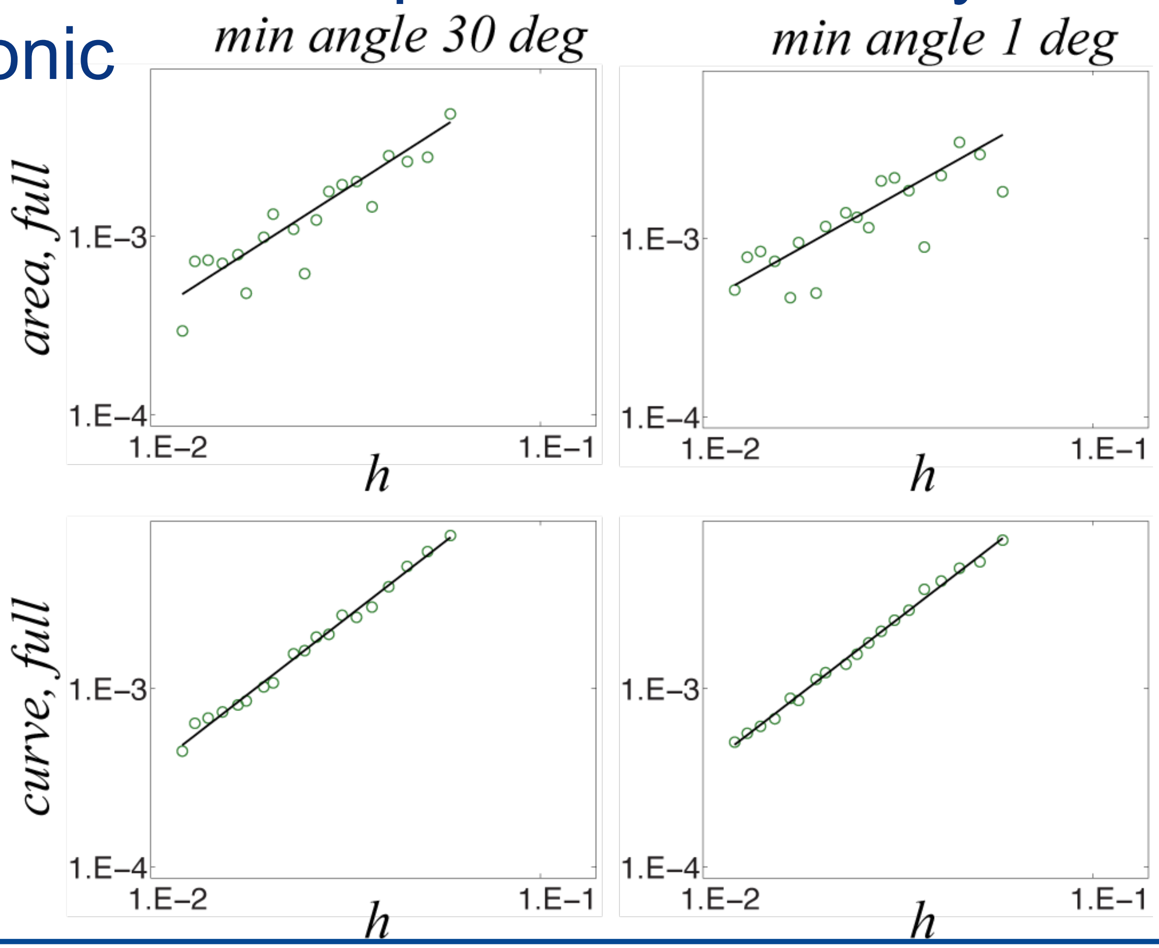
Nearly optimal convergence for biharmonic



Experimental Results

$$\Delta^3 \mathbf{u} = 0$$

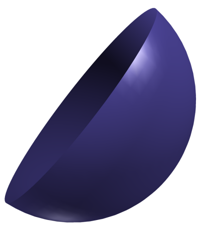
Boundary conditions perform differently for triharmonic



Applications

$$\Delta^2 \mathbf{u} = 0$$

Filling in holes: Laplacian energy

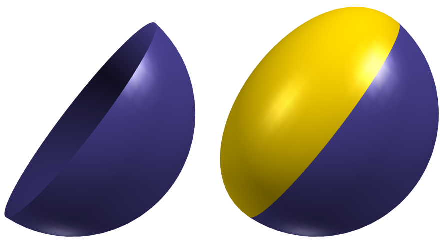


input

Applications

$$\Delta^2 \mathbf{u} = 0$$

Filling in holes: Laplacian energy



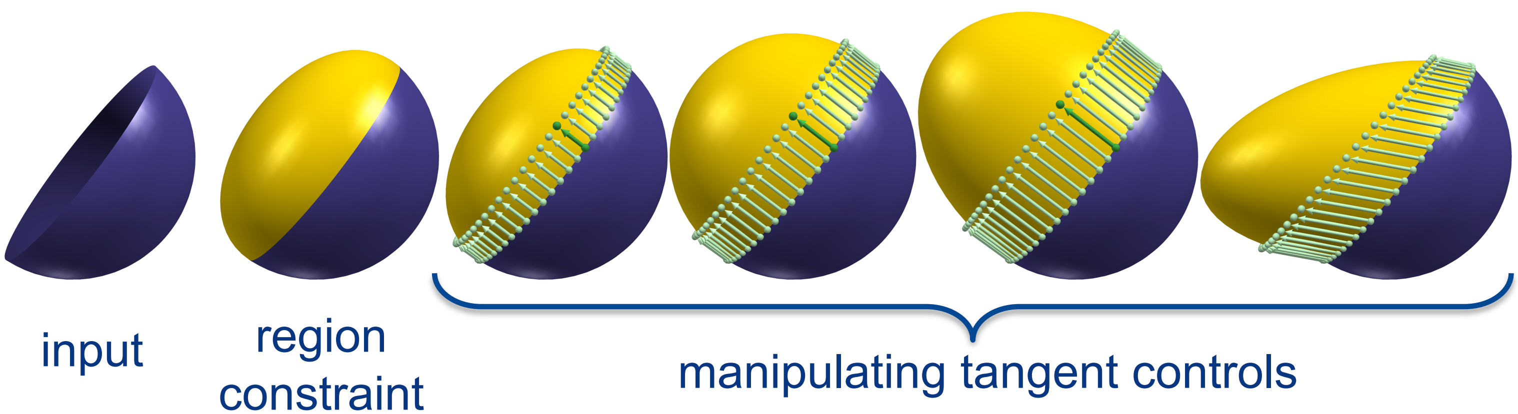
input

region constraint

Applications

$$\Delta^2 \mathbf{u} = 0$$

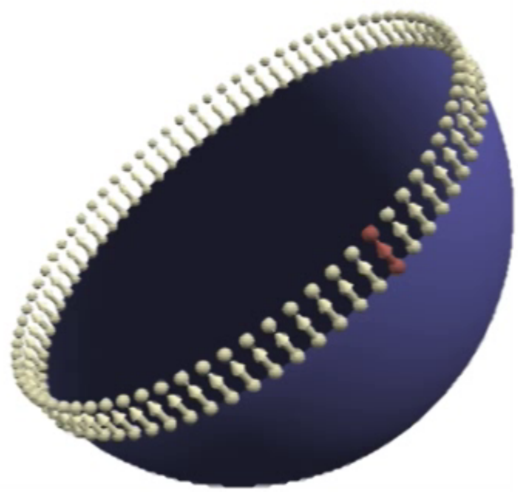
Filling in holes: Laplacian energy



Applications

$$\Delta^2 \mathbf{u} = 0$$

Filling in holes: Laplacian energy

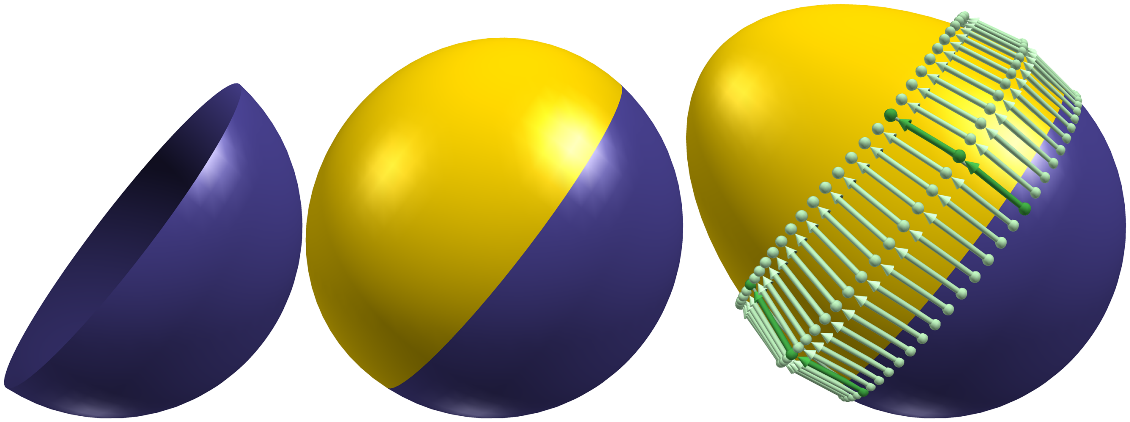


manipulating tangent controls

Applications

$$\Delta^3 \mathbf{u} = 0$$

Filling in holes: Laplacian gradient energy



input

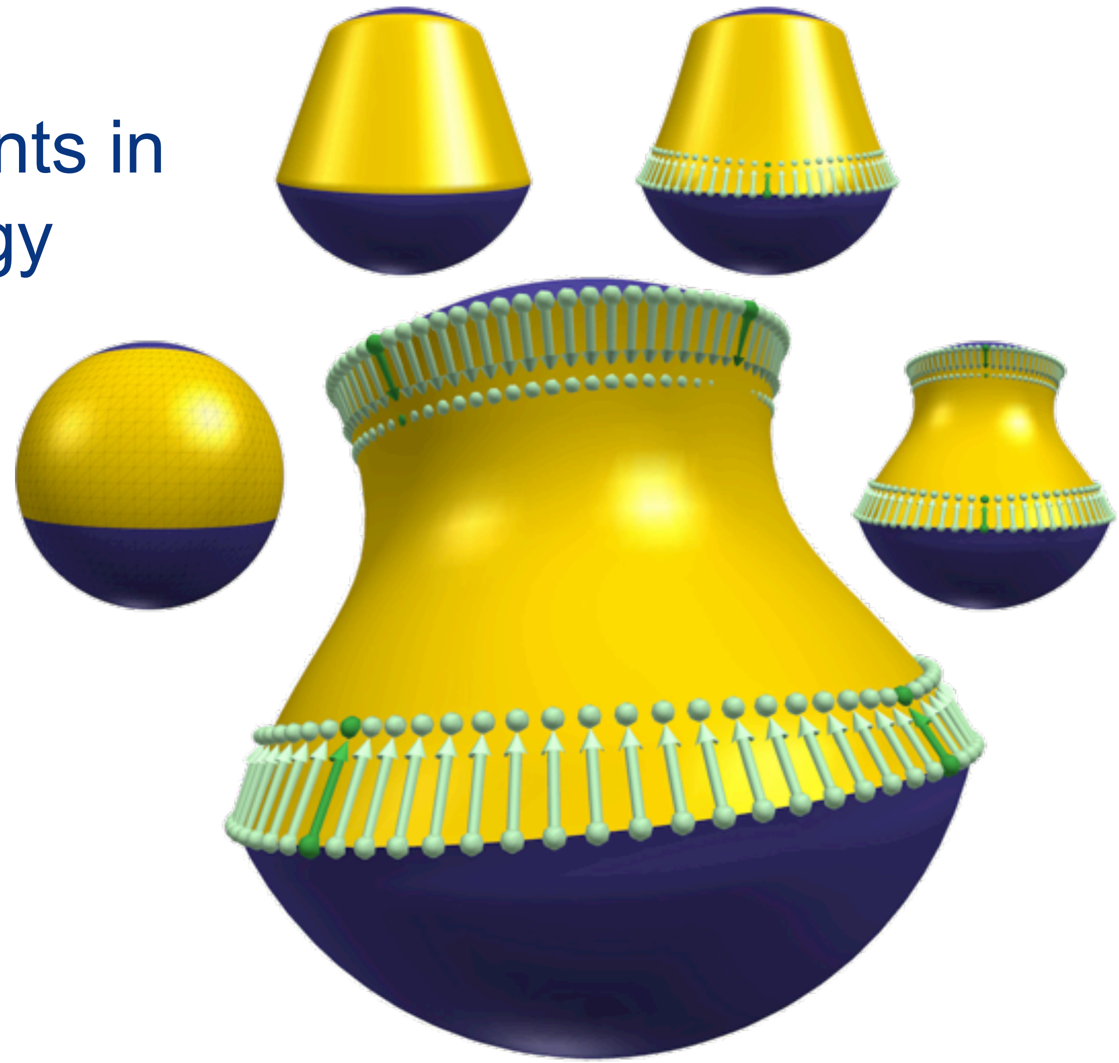
region constraint

manipulating curvature controls

Applications

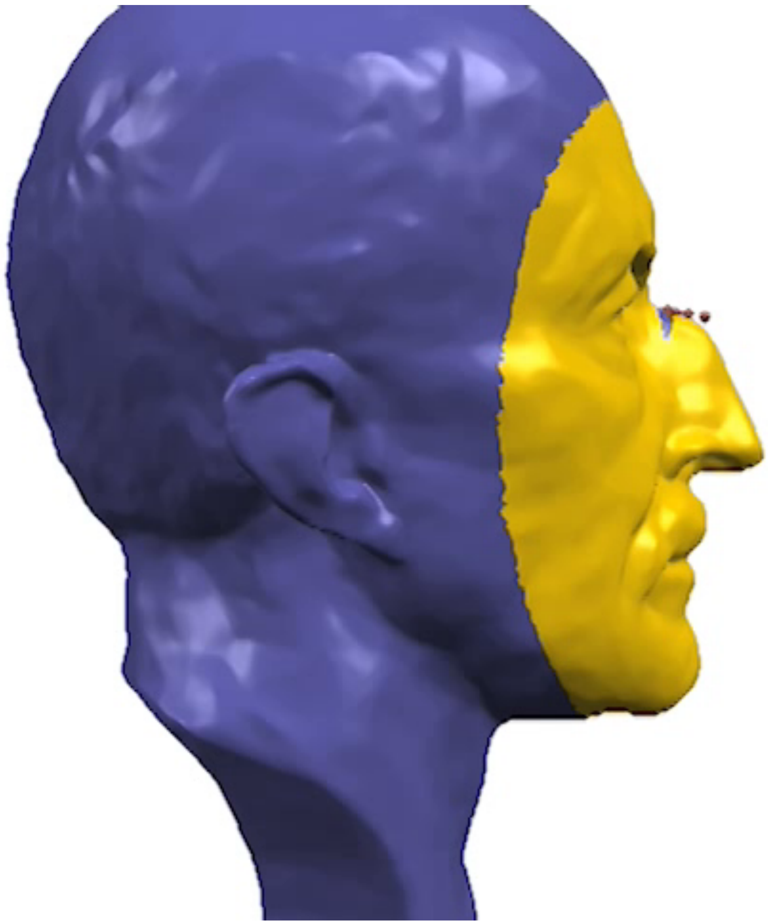
$$\Delta^2 \mathbf{u} = 0$$

- Specifying tangents in Laplacian energy around regions

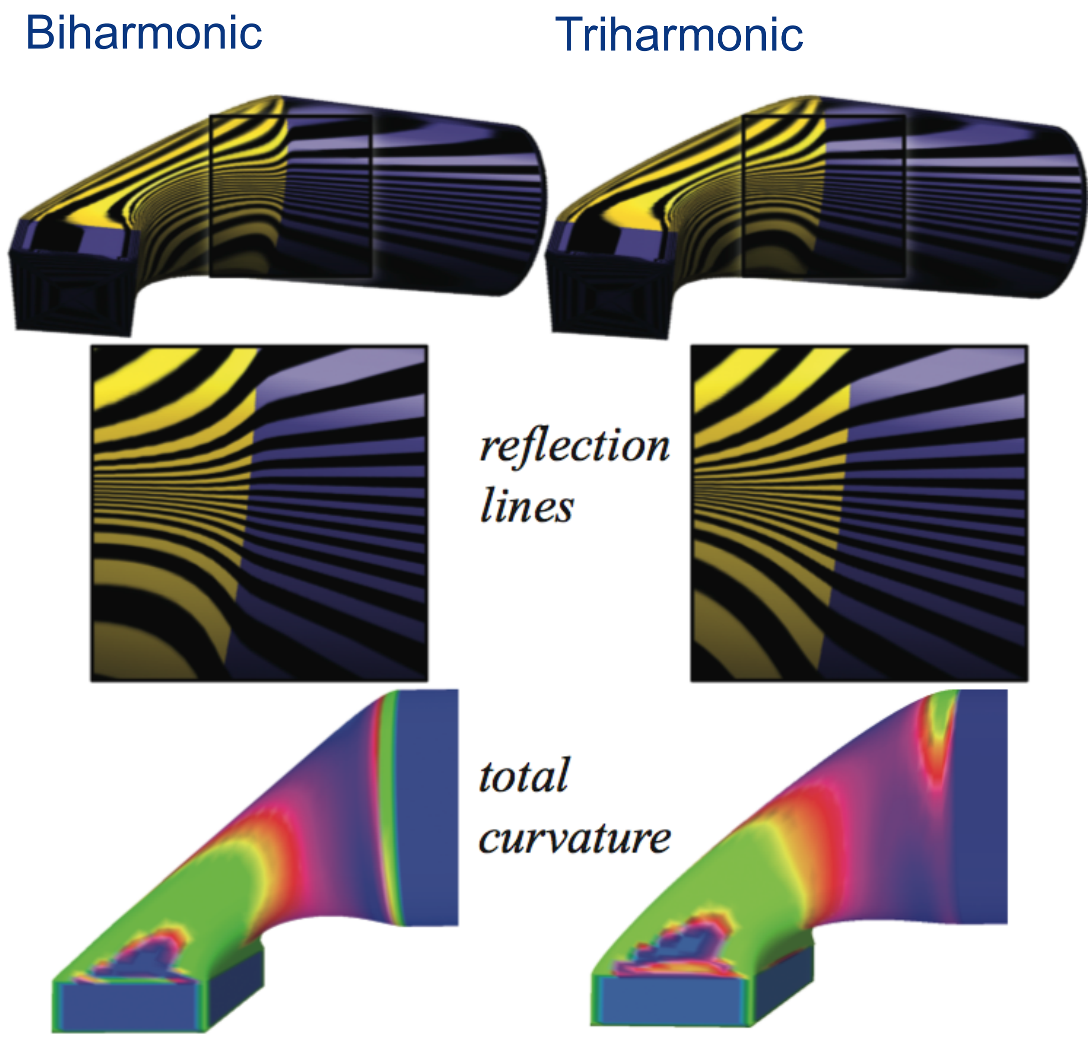


Applications

$$\Delta^3 \mathbf{u} = 0$$



Applications



Summary

Technique for discretizing energies or PDEs

- Reduce to low order by introducing variables
- Use constraints to enforce region boundary conditions
- Lump mass matrix

Convergence for fourth- and sixth-order PDEs

Summary

Technique for discretizing energies or PDEs

- Reduce to low order by introducing variables
- Use constraints to enforce region boundary conditions
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Convergence for fourth- and sixth-order PDEs

Future work

- Improve convergence of triharmonic solution
- Explore using non-flat metric

Acknowledgement of funding

This work was supported in part by an NSF award IIS- 0905502.

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