Smooth Shape-Aware Functions with Controlled Extrema

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INTERACTIVE GEOMETRY LAB

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Swiss Federal Institute of Technology Zurich

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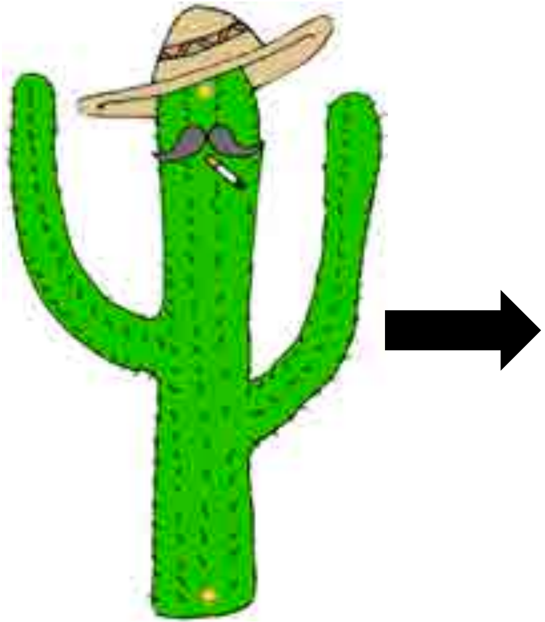
Real-time deformation relies on smooth, shape-aware functions

input shape + handles



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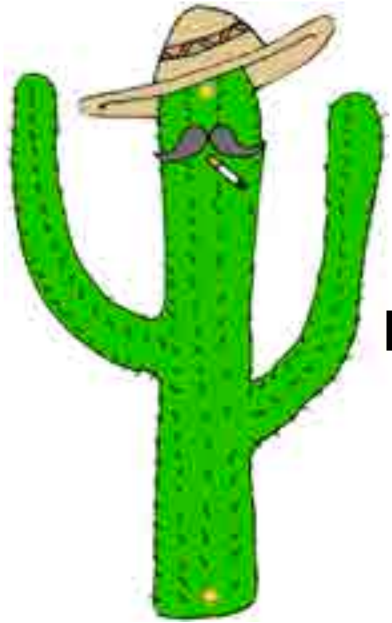
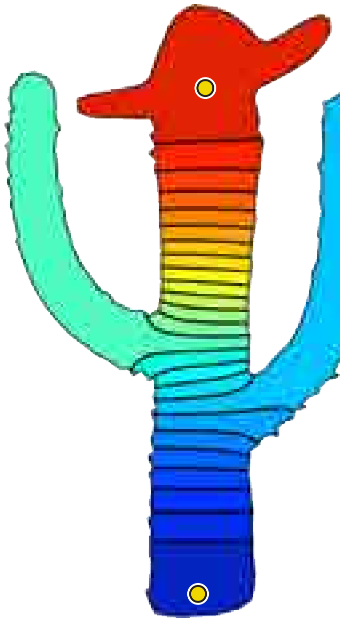
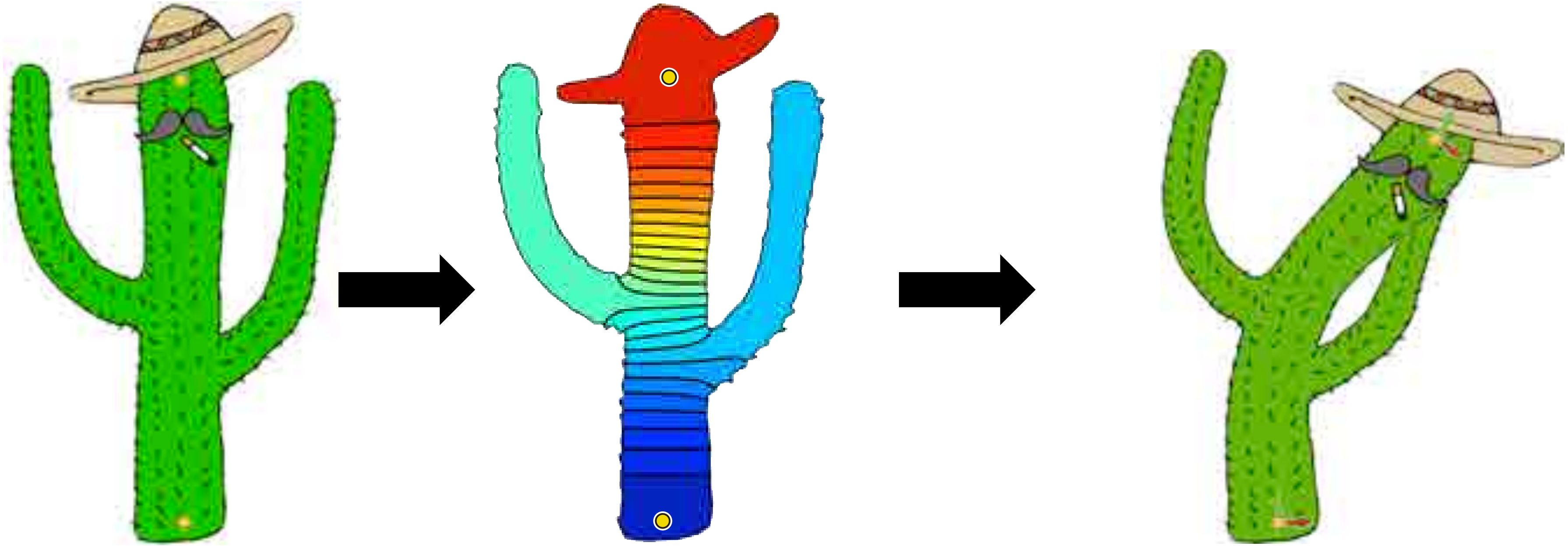
precompute weight functions





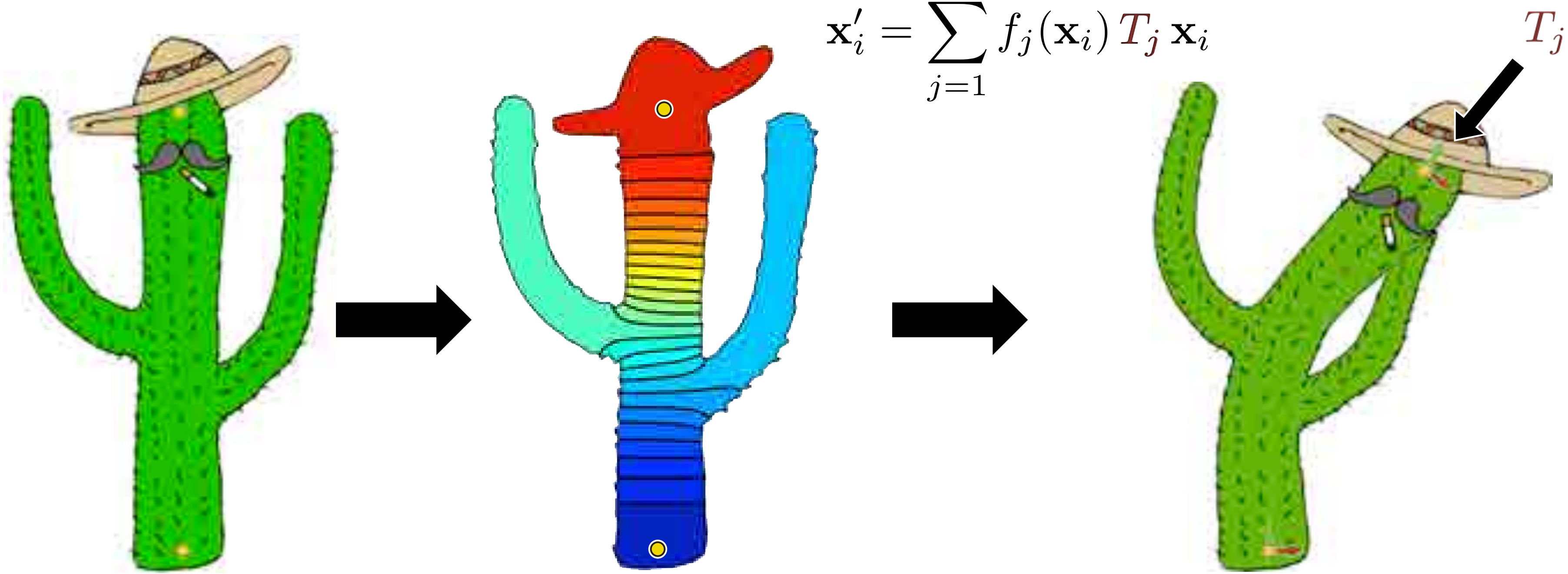
Real-time deformation relies on smooth, shape-aware functions

deform handles → deform shape



Real-time deformation relies on smooth, shape-aware functions

$$\mathbf{x}'_i = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$



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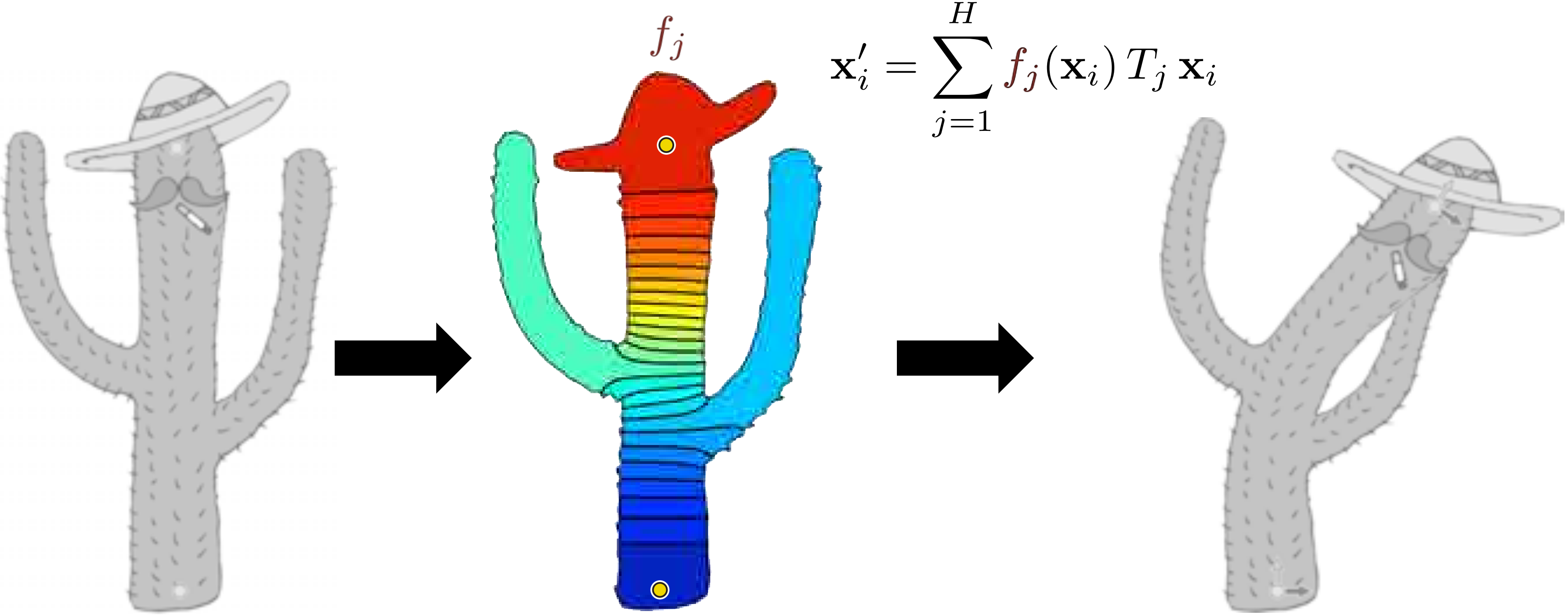
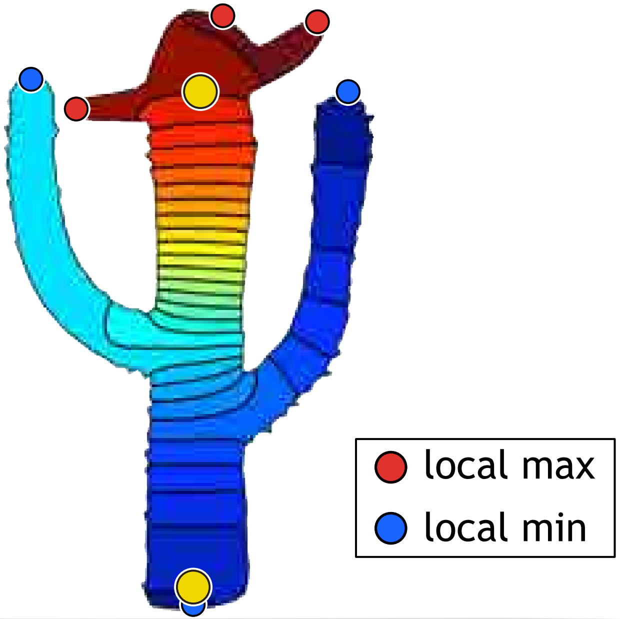


$$\mathbf{x}'_i = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$

Spurious extrema cause distracting artifacts

unconstrained Δ^2
[Botsch & Kobbelt 2004]



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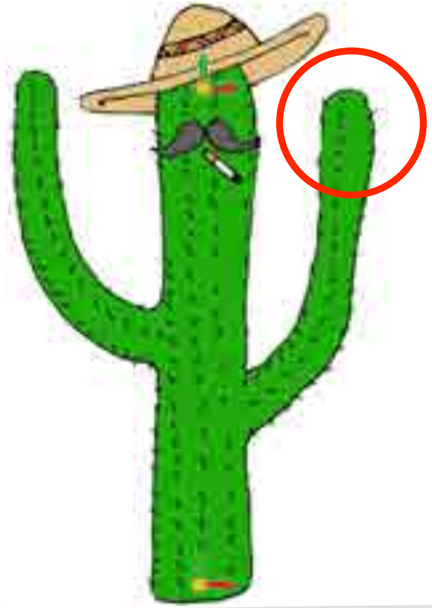

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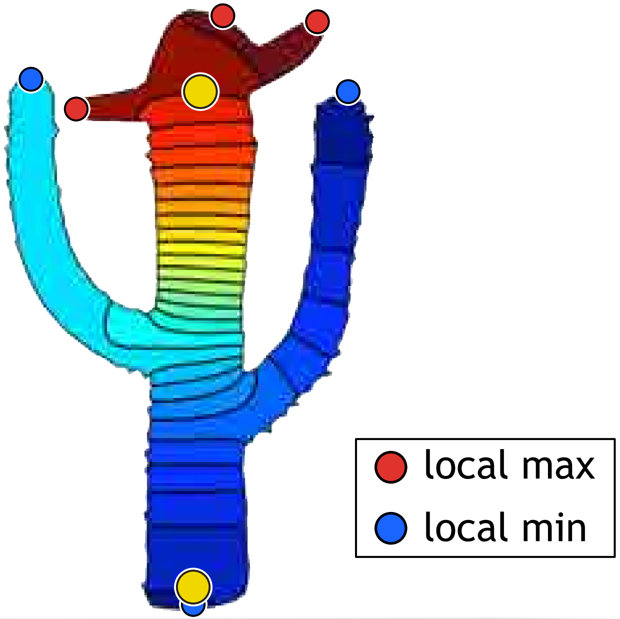
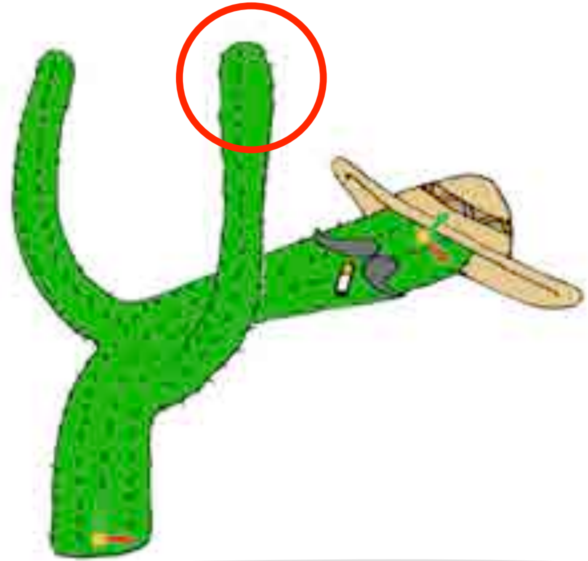
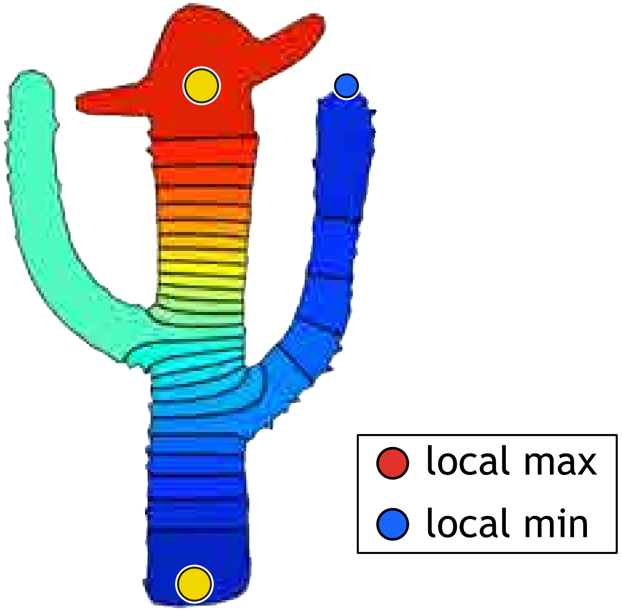


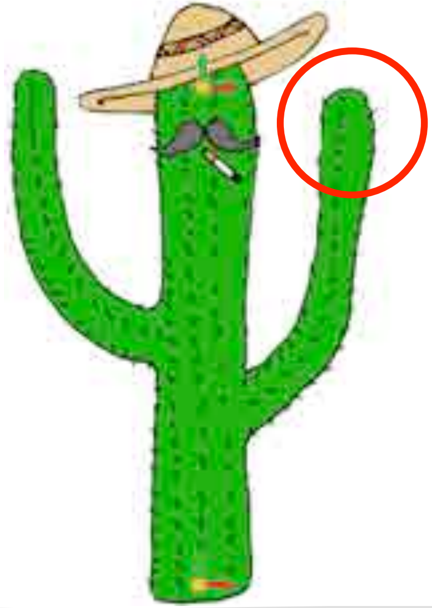


Bounds help, but don't solve problem

bounded Δ^2
[Jacobson et al. 2011]

$$\mathbf{x}_i' = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$





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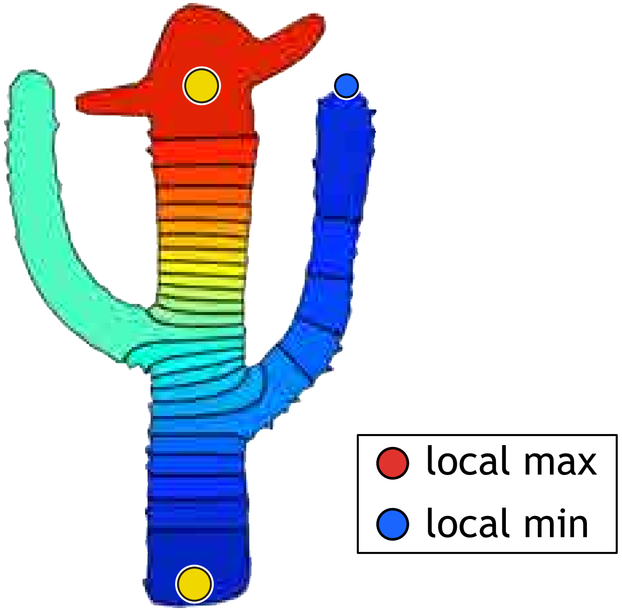

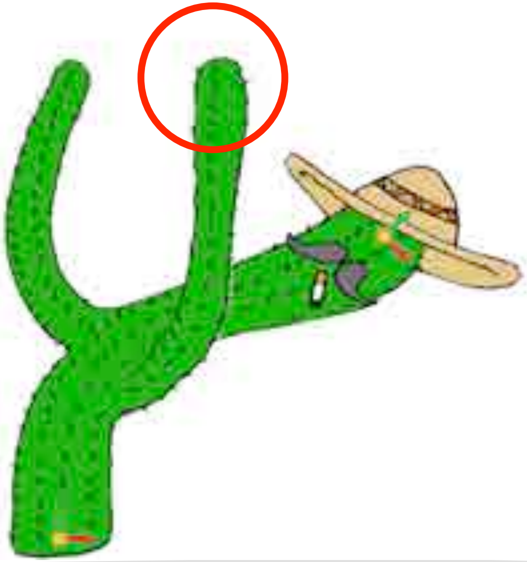

Gets worse with higher-order smoothness

bounded Δ^4
[Jacobson et al. 2011]

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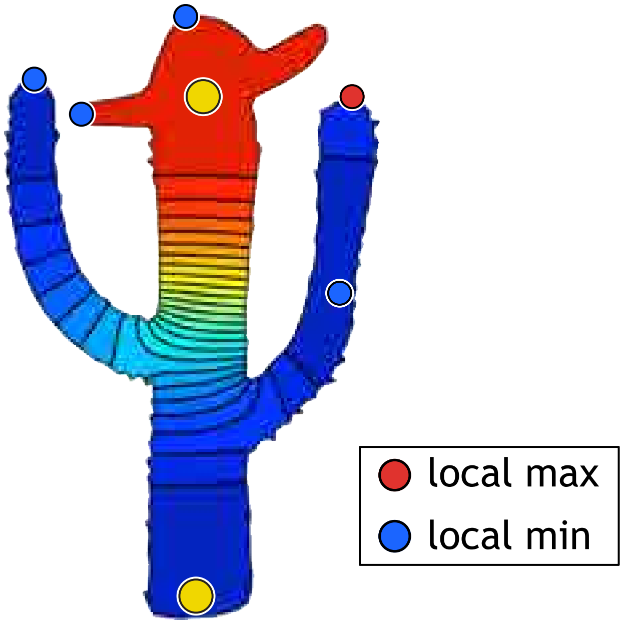

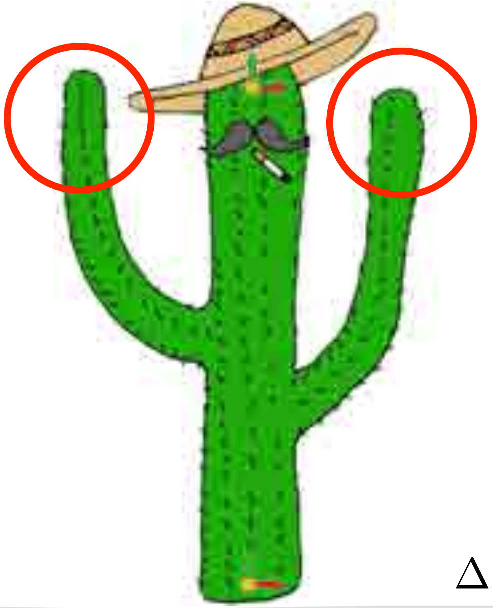

$\Delta^k,\ k > 2$ *oscillate too much*

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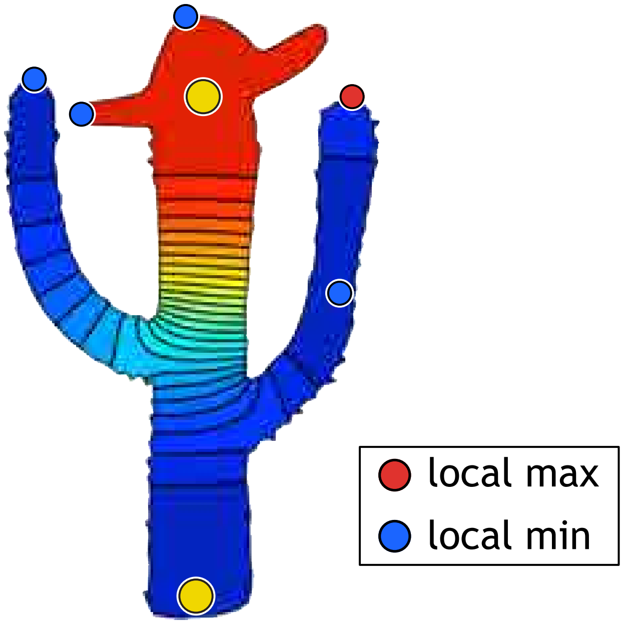

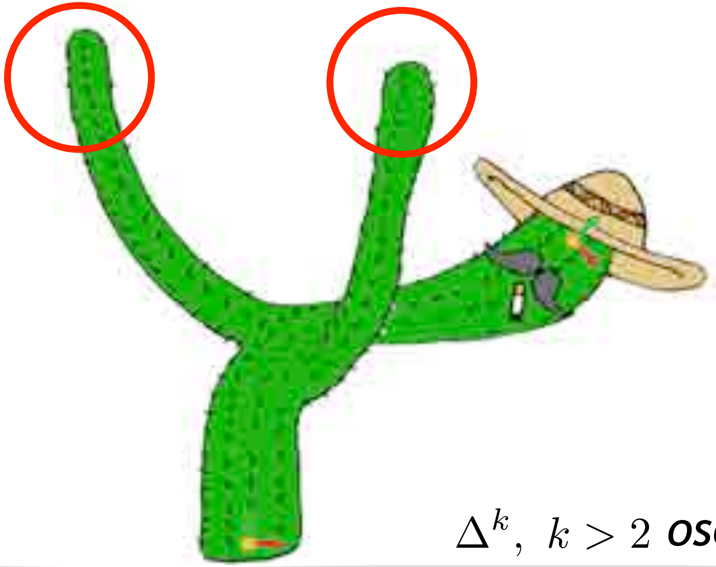

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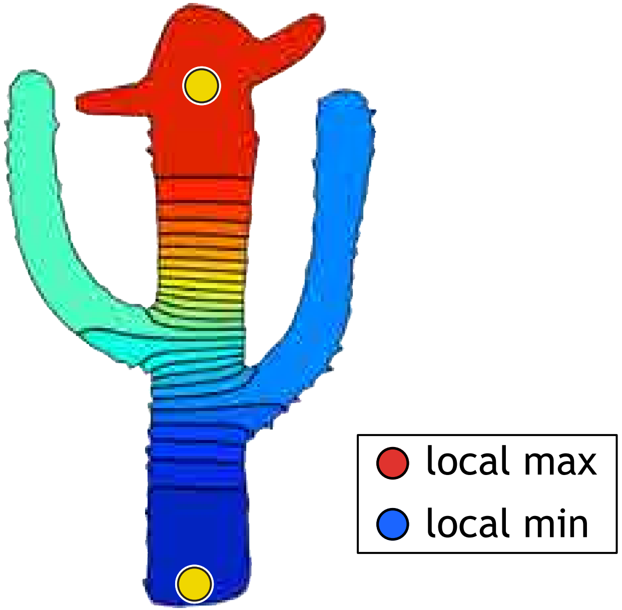
We *explicitly* prohibit spurious extrema

our Δ^4



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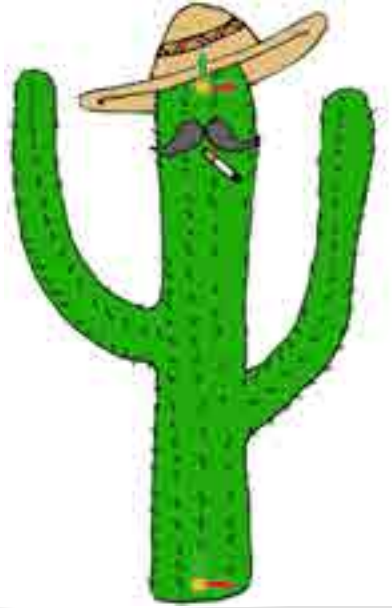

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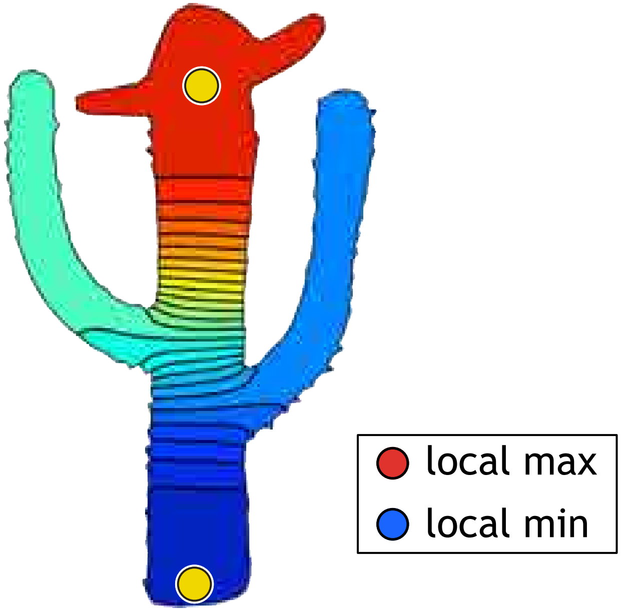

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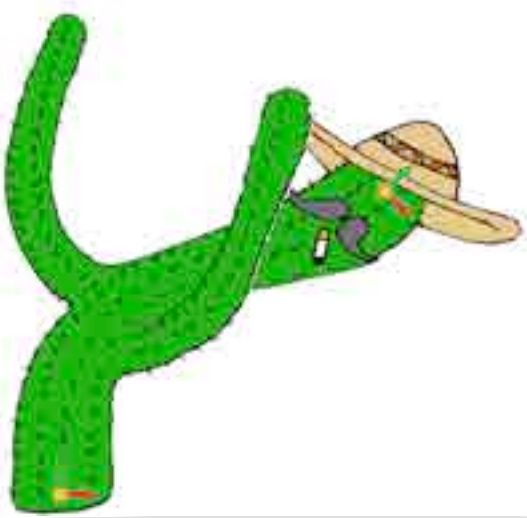

Same functions used for color interpolation

$$\mathbf{x}_i' = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$

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Same functions used for color interpolation

$$\mathbf{c}_i = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\,\mathbf{c}_j$$

Same functions used for color interpolation

unconstrained Δ^2
[Finch et al. 2011]

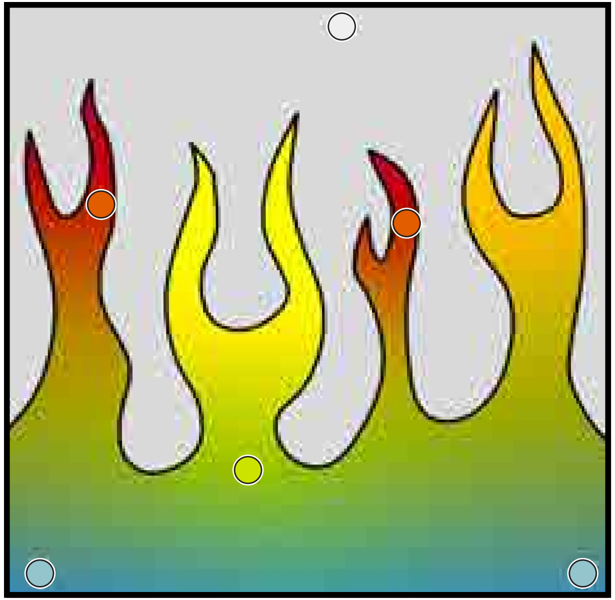

Image courtesy Mark Finch

$$\mathbf{c}_i = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\mathbf{c}_j$$

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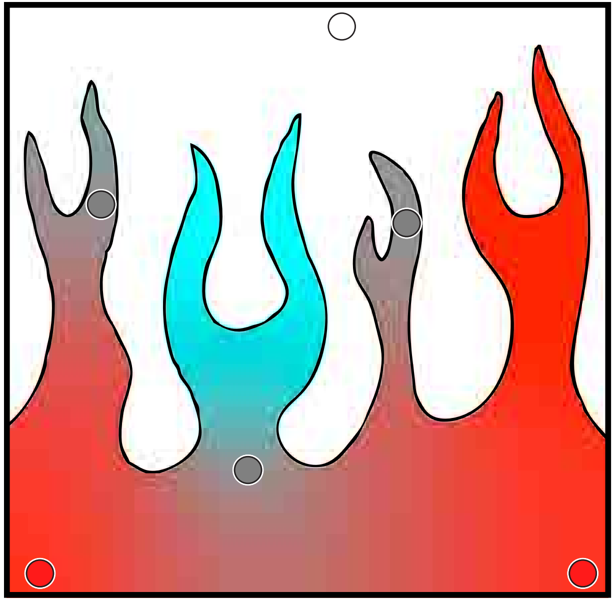

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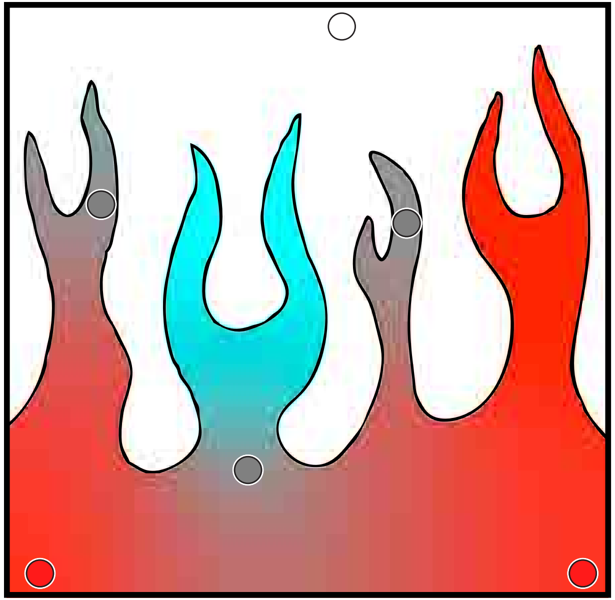

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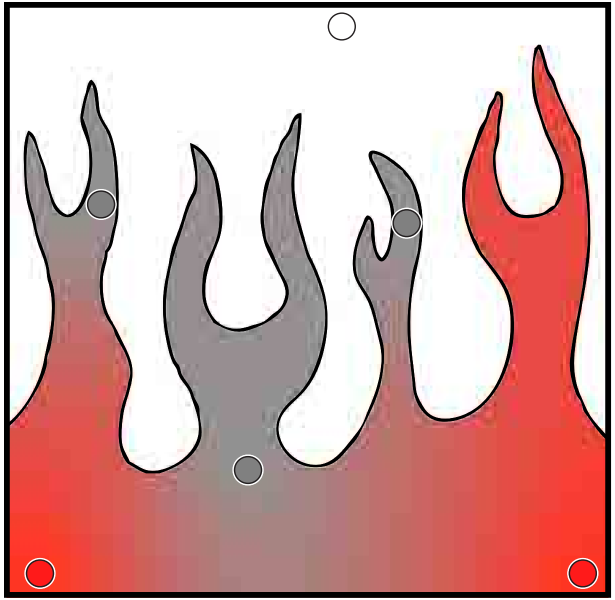
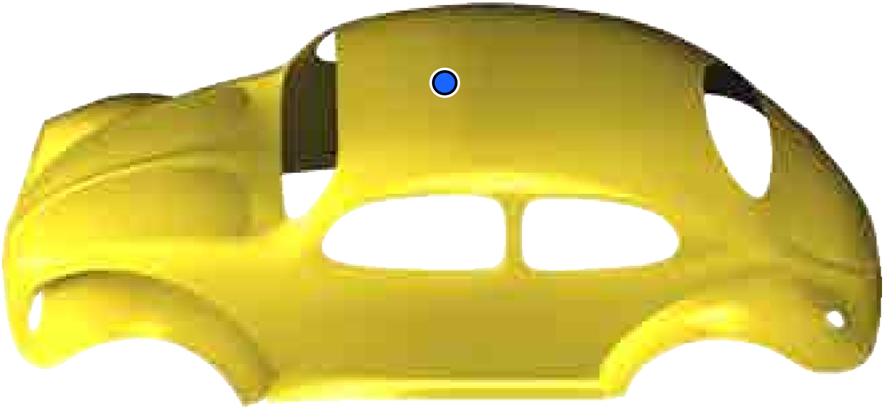


Want same control when smoothing data



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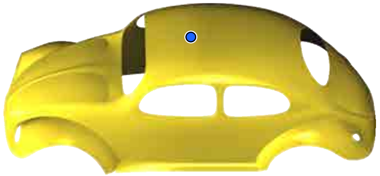

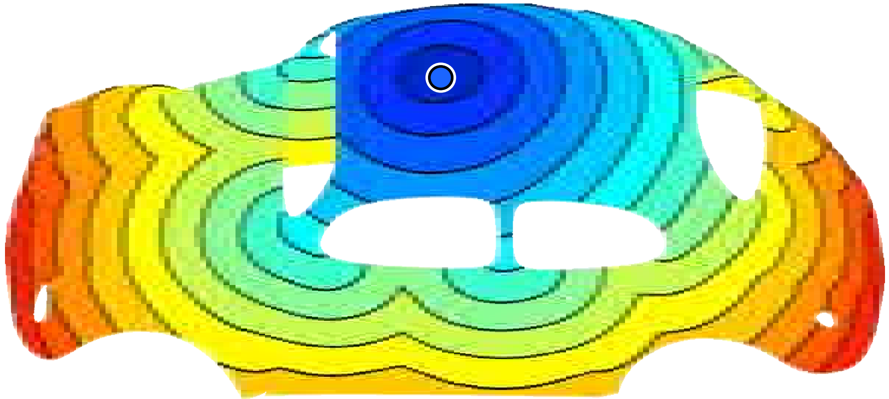

Exact, but sharp geodesic

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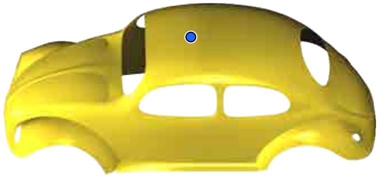

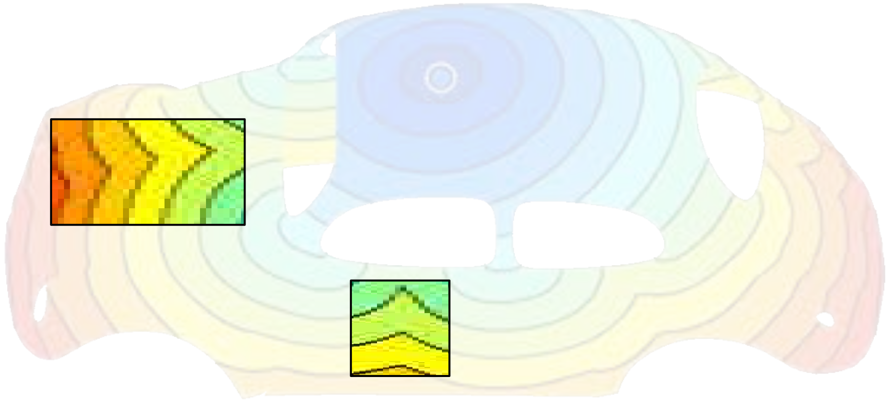

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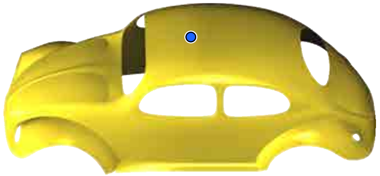

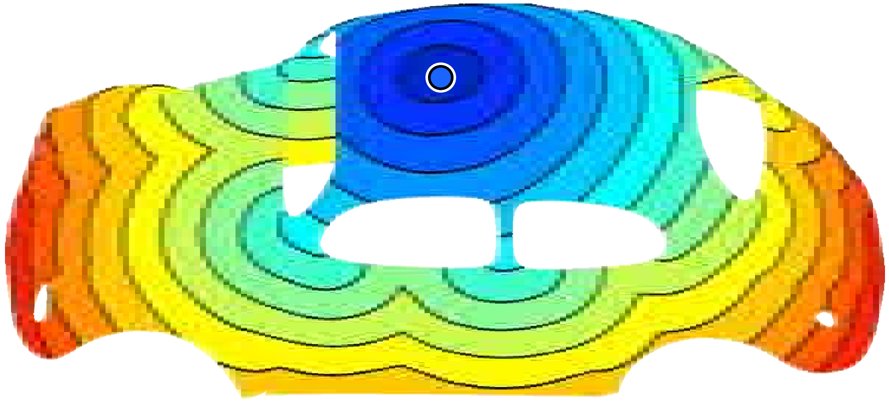

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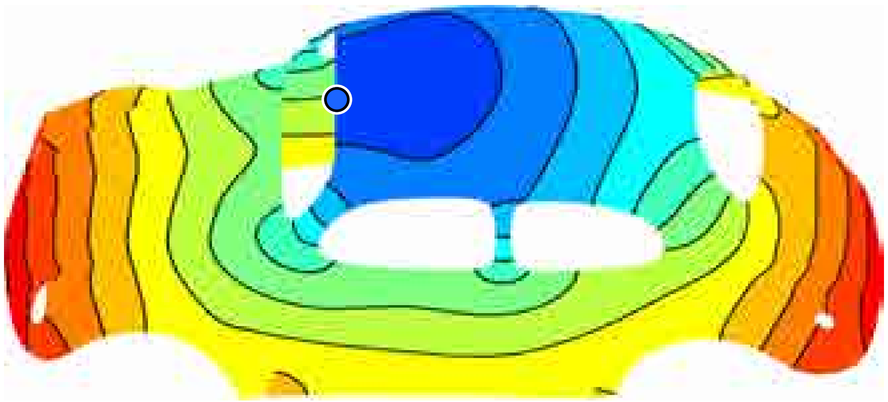
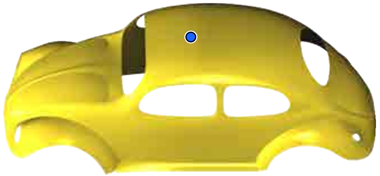
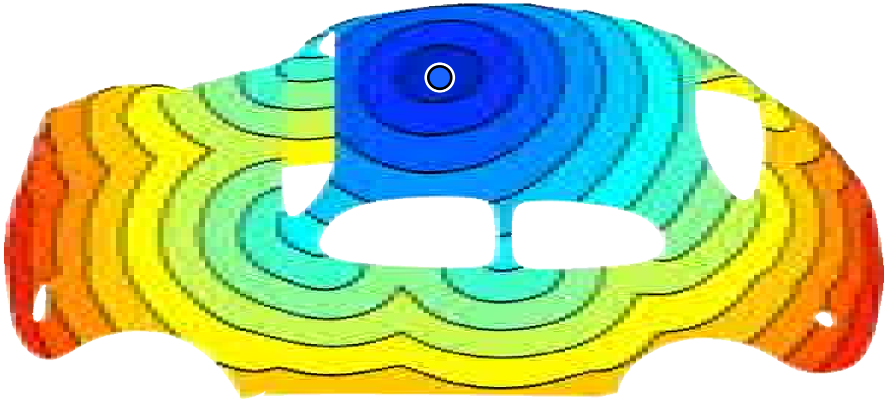


Smooth, but extrema are lost

Want same control when smoothing data





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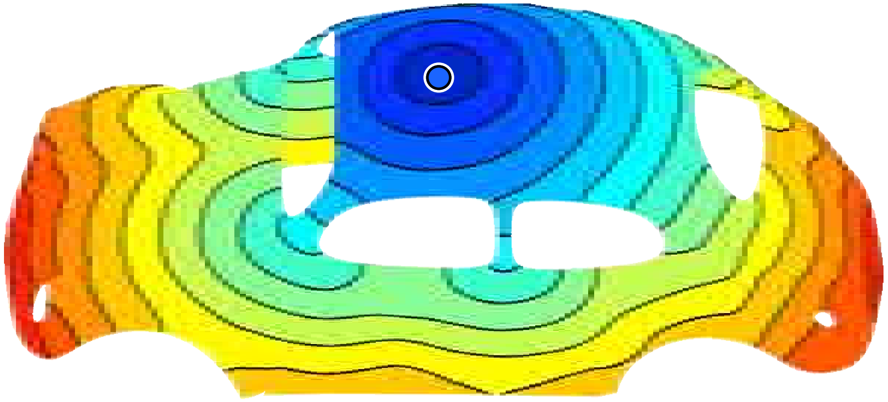


Smooth *and* maintain extrema

Ideal discrete problem is intractable

$$\arg\min_{f} \; E(f)$$

Interpolation functions:

$$E_L(f) = \int_{\mathcal{M}} \|\nabla^k f\|^2 dV, \quad k = 2, 3, \ldots$$

Ideal discrete problem is intractable

$$\arg\min_f \; E(f)$$

Data smoothing:

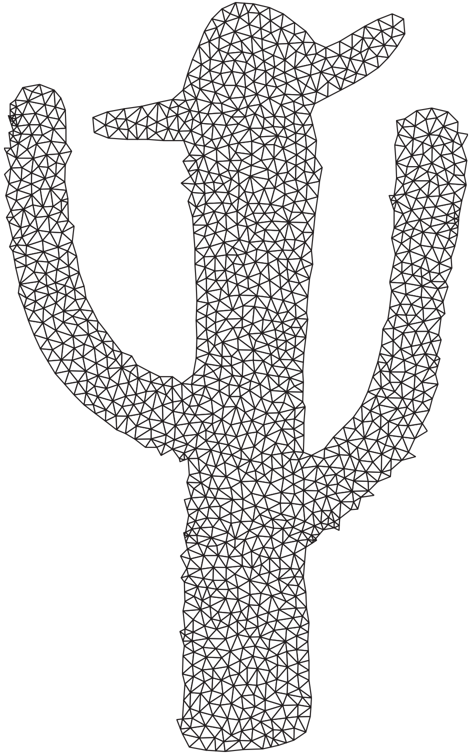
$$E_L(f) = \int_{\mathcal{M}} \|\nabla^k f\|^2 dV, \quad k = 2, 3, \dots$$

$$E_D(f) = \sum_{i \in \mathcal{M}} \|h_i - f_i\|^2$$

$$E(f) = \gamma_L E_L(f) + \gamma_D E_D(f)$$

Ideal discrete problem is intractable

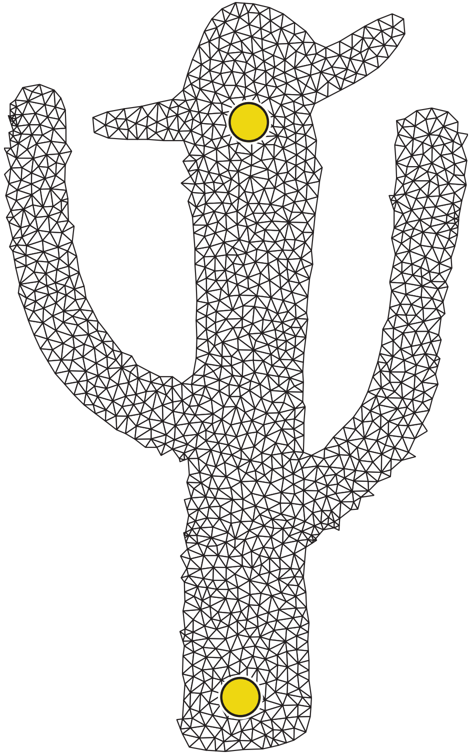
$$\arg\min_f \; E(f)$$



Ideal discrete problem is intractable

$$\arg\min_{f} \; E(f)$$

$$\text{s.t.} \quad f_{\max} = known$$
$$f_{\min} = known$$



Ideal discrete problem is intractable

$$\arg\min_f \; E(f)$$

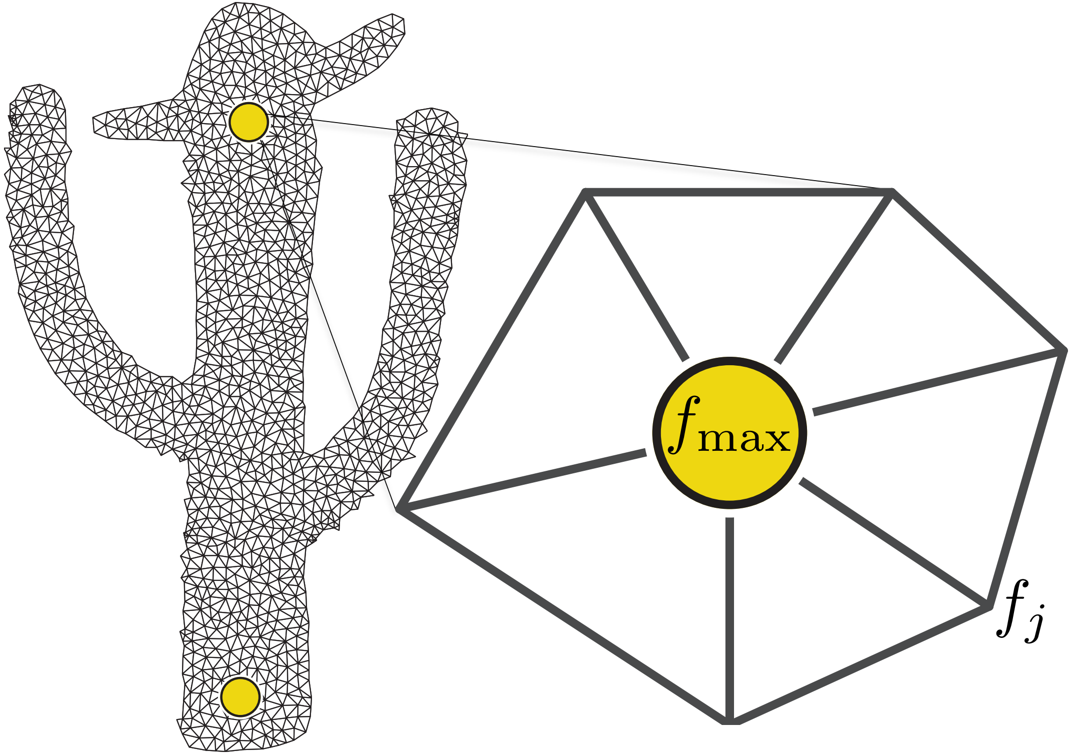
$$\text{s.t.} \quad f_{\max} = known$$

$$f_{\min} = known$$

linear

$$f_j < f_{\max}$$

$$f_j > f_{\min}$$



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linear

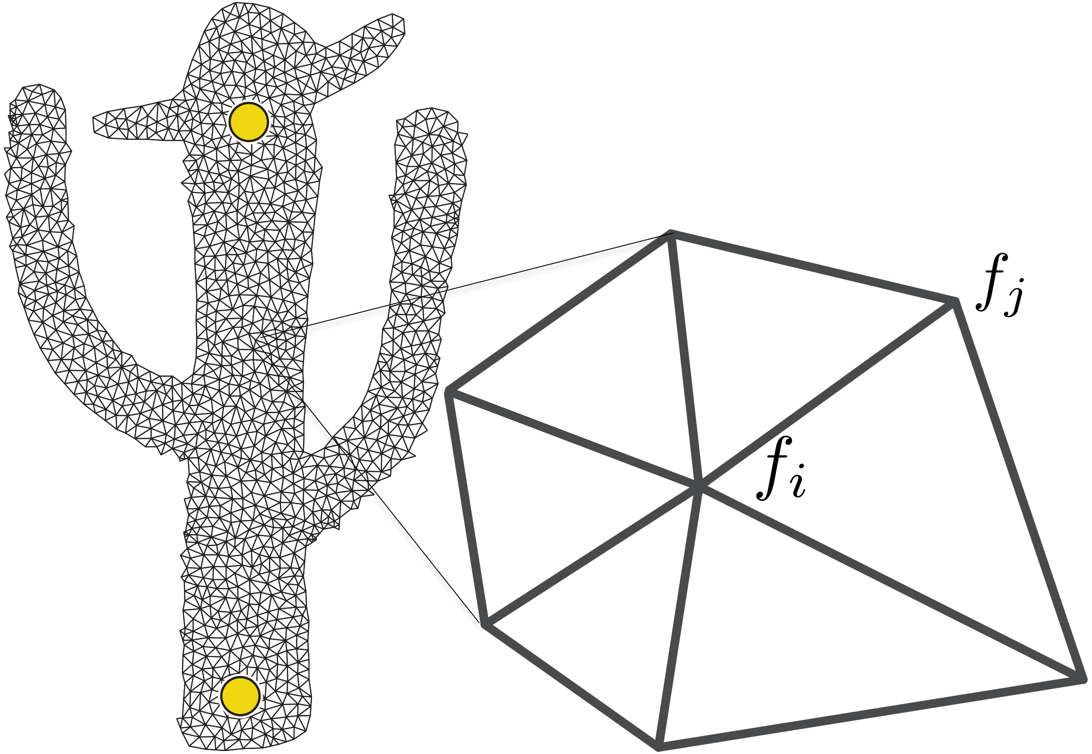
$$f_j < f_{\max}$$

$$f_j > f_{\min}$$

nonlinear

$$f_i > \min_{j \in \mathcal{N}(i)} f_j$$

$$f_i < \max_{j \in \mathcal{N}(i)} f_j$$



Assume we have a feasible solution

$$\arg\min_f \; E(f)$$

	s.t. $f_{\max} = known$
	$f_{\min} = known$
linear	$f_j < f_{\max}$
	$f_j > f_{\min}$
nonlinear	$f_i > \min_{j \in \mathcal{N}(i)} f_j$
	$f_i < \max_{j \in \mathcal{N}(i)} f_j$

"Representative function" u

$u_j < u_{\max}$	handles
$u_j > u_{\min}$	
$u_i > \min_{j \in \mathcal{N}(i)} u_j$	interior
$u_i < \max_{j \in \mathcal{N}(i)} u_j$	

Assume we have a feasible solution

"Representative function" u

handles	$u_j < u_{\max}$ $u_j > u_{\min}$
interior	$u_i > \min_{j \in \mathcal{N}(i)} u_j$ $u_i < \max_{j \in \mathcal{N}(i)} u_j$

Copy "monotonicity" of representative

$$\arg\min_{f} \; E(f)$$

$$\text{s.t.} \quad f_{\max} = known$$

$$f_{\min} = known$$

$$(f_i - f_j)(u_i - u_j) > 0 \qquad \text{linear} \qquad \forall (i,j) \in \mathcal{E}$$

At least one edge in either direction per vertex

Rewrite as conic optimization

QP

$$\begin{aligned}\underset{\mathbf{f}}{\text{minimize}} \quad & \frac{1}{2}\|\mathbf{F}\,\mathbf{f}\|^2 + \mathbf{c}^\mathsf{T}\mathbf{f} + \text{const}\\ \text{subject to} \quad & \mathbf{A}_{leq}^\mathsf{T}\mathbf{f} \le \mathbf{b}_{leq},\\ & \mathbf{f} \le \mathbf{u}_f, \quad \mathbf{f} \ge \mathbf{l}_f\end{aligned}$$

→

Conic

$$\begin{aligned}\underset{\begin{bmatrix}\mathbf{f}^\mathsf{T} & \mathbf{t}^\mathsf{T} & v\end{bmatrix}}{\text{minimize}} \quad & \begin{bmatrix}\mathbf{c}^\mathsf{T} & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{f}\\ \mathbf{t}\\ v\end{bmatrix} + \text{const}\\ \text{subject to} \quad & \begin{bmatrix}\mathbf{F} & -\mathbf{I} & 0\\ \mathbf{A}_{leq}^\mathsf{T} & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{f}\\ \mathbf{t}\\ v\end{bmatrix} \ge \begin{bmatrix}0\\ -\infty\end{bmatrix}\\ & \begin{bmatrix}\mathbf{F} & -\mathbf{I} & 0\\ \mathbf{A}_{leq}^\mathsf{T} & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{f}\\ \mathbf{t}\\ v\end{bmatrix} \le \begin{bmatrix}0\\ \mathbf{b}_{leq}\end{bmatrix}\\ & \begin{bmatrix}\mathbf{f}\\ \mathbf{t}\\ v\end{bmatrix} \le \begin{bmatrix}\mathbf{u}_f\\ \infty\\ \infty\end{bmatrix}\\ & \begin{bmatrix}\mathbf{f}\\ \mathbf{t}\\ v\end{bmatrix} \ge \begin{bmatrix}\mathbf{l}_f\\ -\infty\\ 0\end{bmatrix}\\ & 2v \ge \sum_i t_i^2\end{aligned}$$

Optimize with MOSEK

We always have harmonic representative

$$\arg\min_u \ \frac{1}{2}\int_\Omega \|\nabla u\|^2 dV$$

We always have harmonic representative

$$\operatorname*{arg\,min}_u \; \frac{1}{2}\int_\Omega \|\nabla u\|^2 dV$$

$$\text{s.t.} \quad u_{\max} = 1$$

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We always have harmonic representative

$$\arg\min_u \frac{1}{2}\int_\Omega \|\nabla u\|^2 dV$$

$$\text{s.t.} \quad u_{\max} = 1$$

$$\text{s.t.} \quad u_{\min} = 0$$

We always have harmonic representative

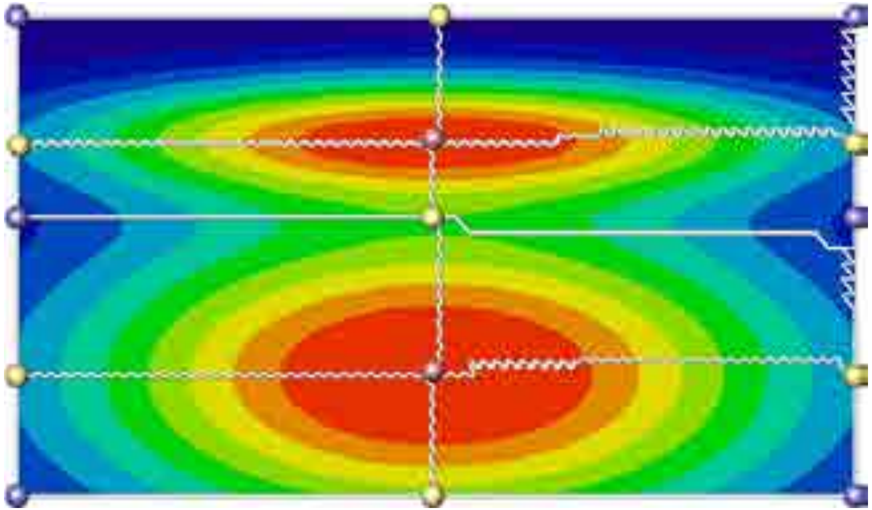
$$\arg\min_{u} \ \frac{1}{2}\int_{\Omega} \|\nabla u\|^2 dV$$

$$\text{s.t.} \quad u_{\max} = 1$$

$$\text{s.t.} \quad u_{\min} = 0$$

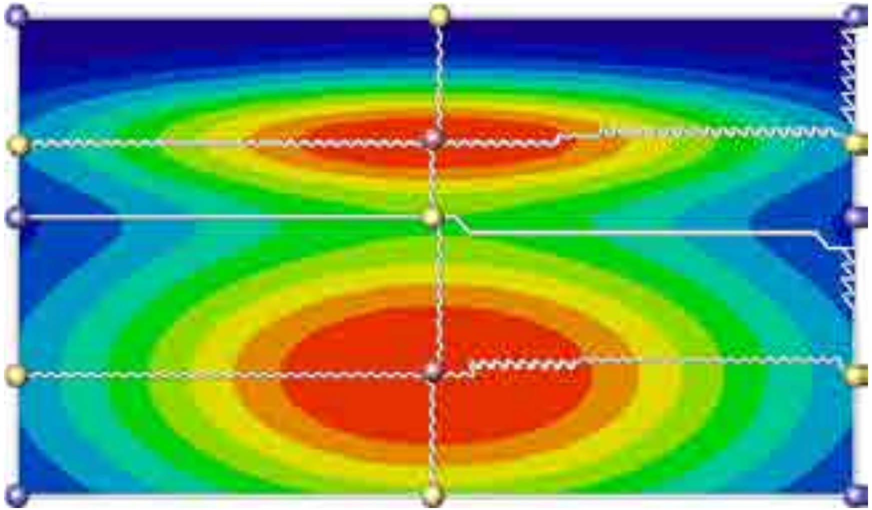
Works well when no input function exists

Data energy may fight harmonic representative



Anisotropic input data

Data energy may fight harmonic representative



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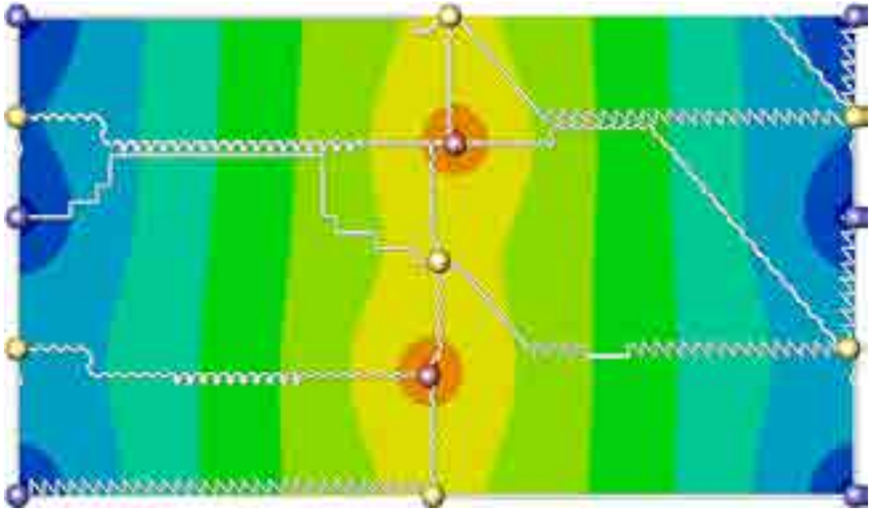

Harmonic representative

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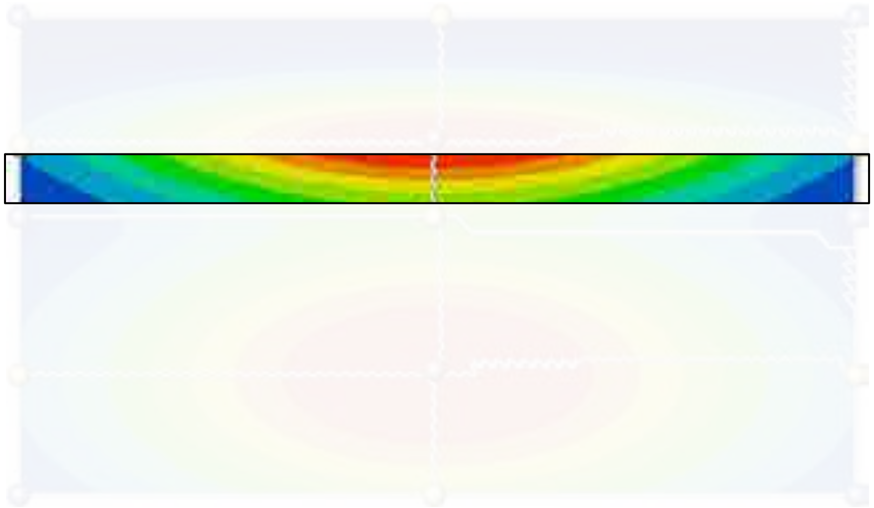

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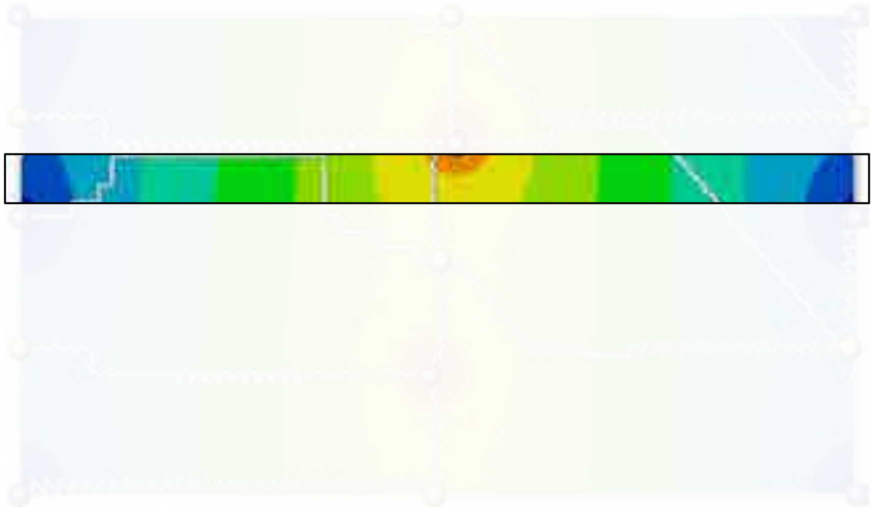

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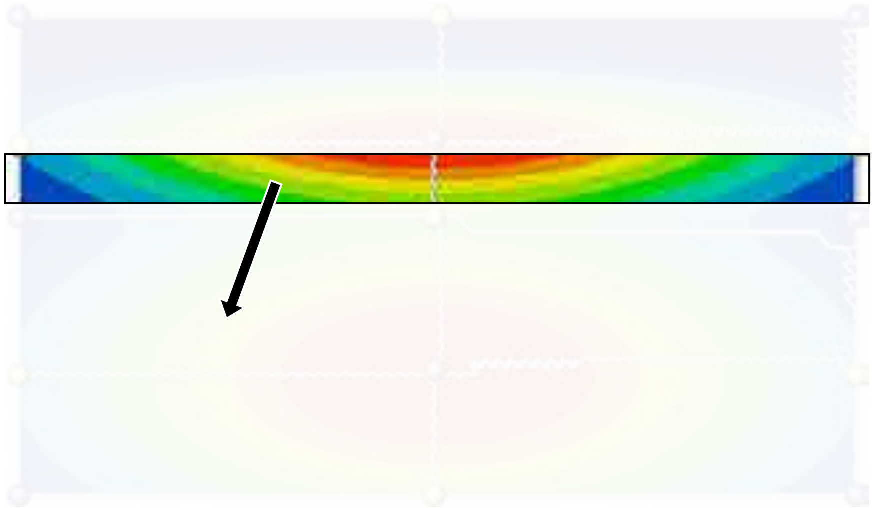

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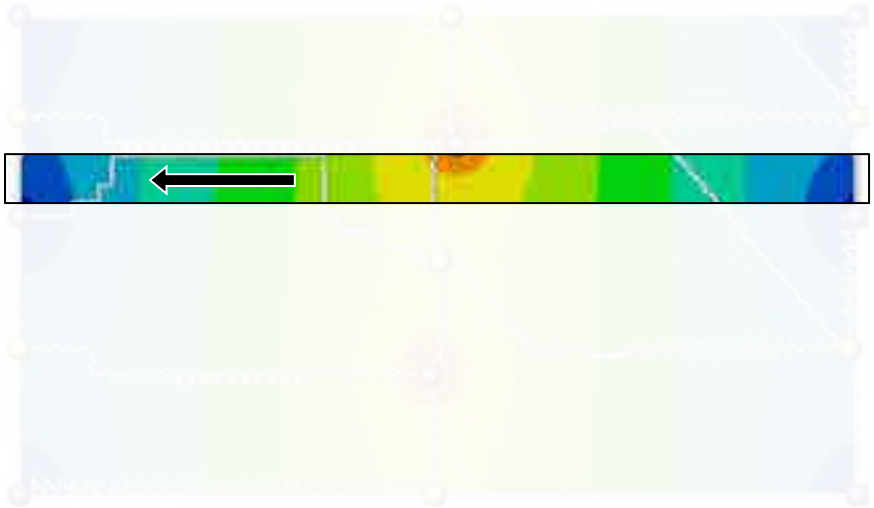

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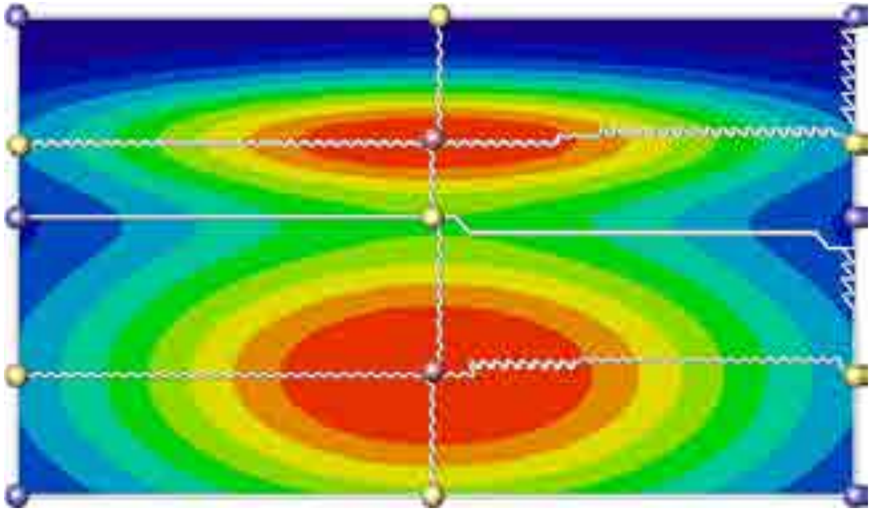

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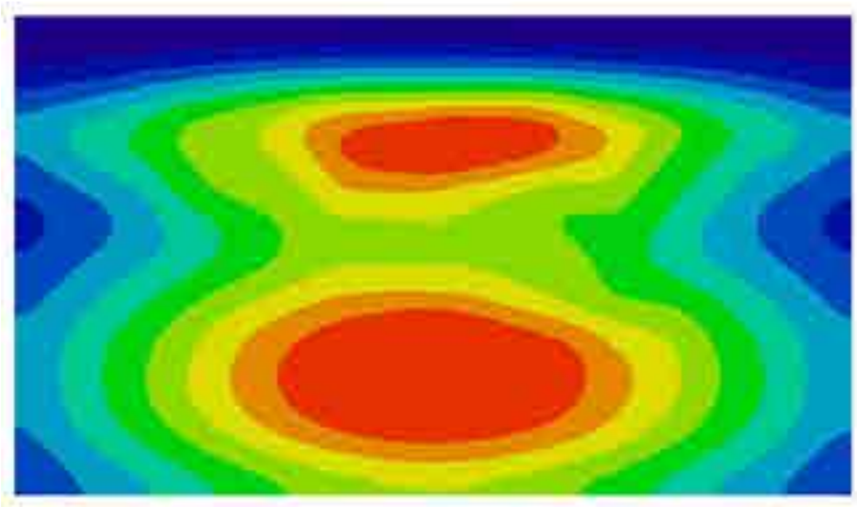
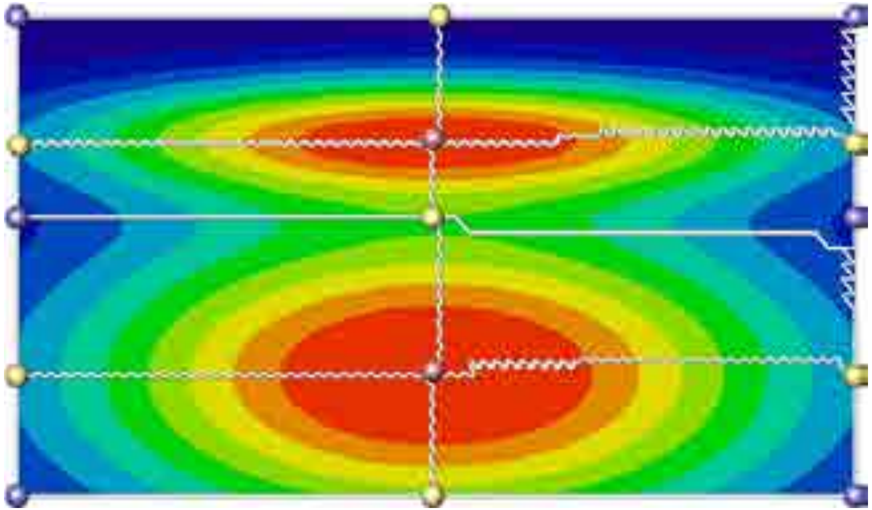
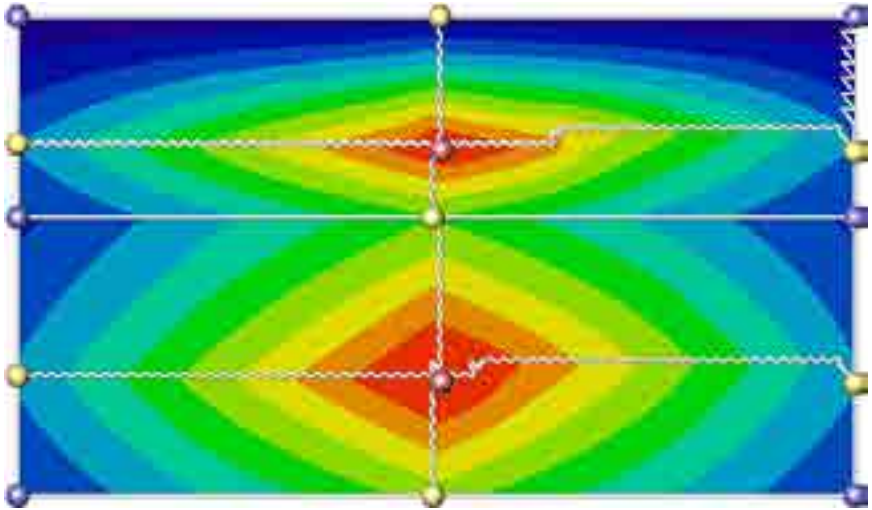


Resulting solution with large γ_D

If data exists, copy topology, too



Anisotropic input data



[Weinkauf et al. 2010]
representative

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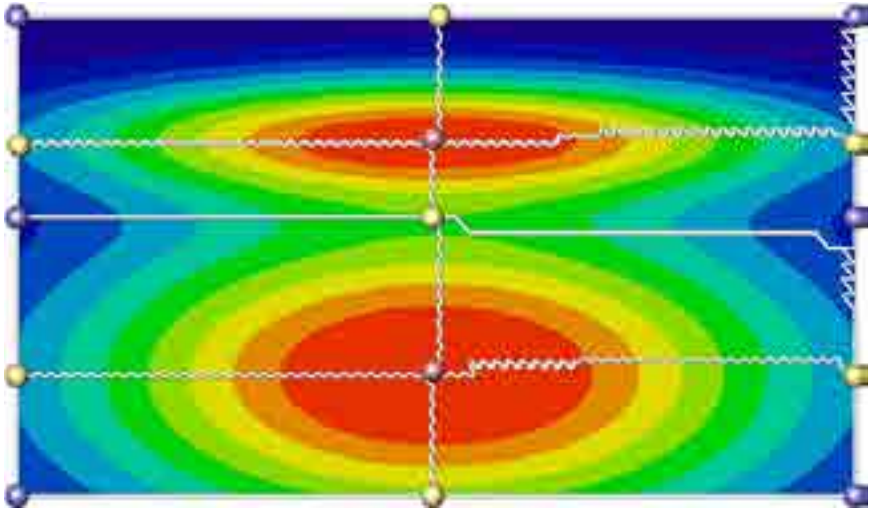

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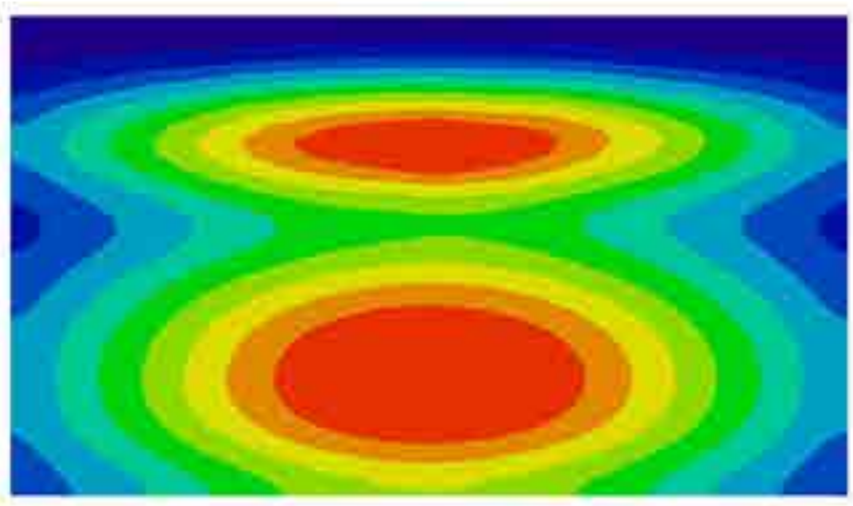

Resulting solution with large γ_D

Final algorithm is simple and efficient

- *Data smoothing*: topology-aware representative
  - Morse-smale + linear solve ~milliseconds

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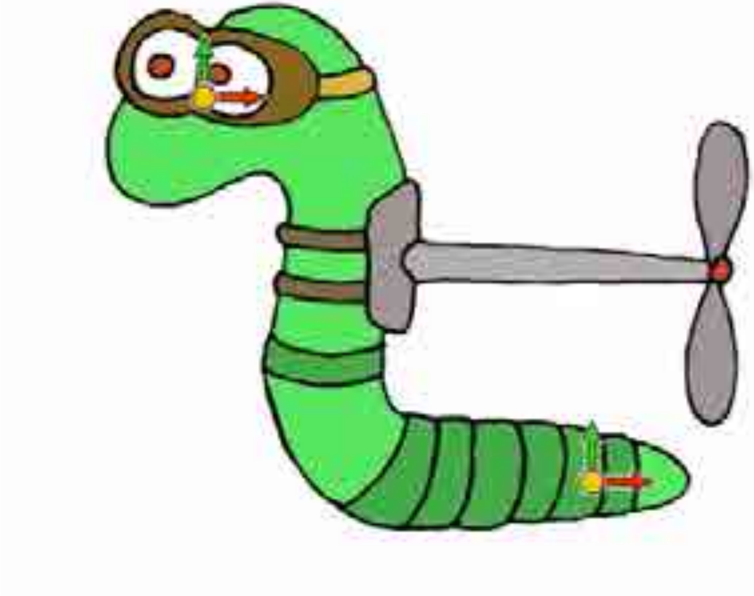
Interpolation: functions are **precomputed**

We preserve troublesome appendages

Bounded Δ^2



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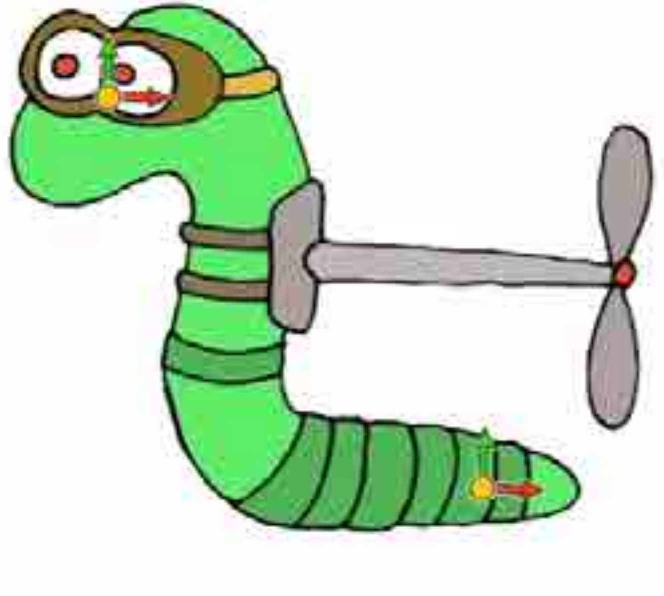

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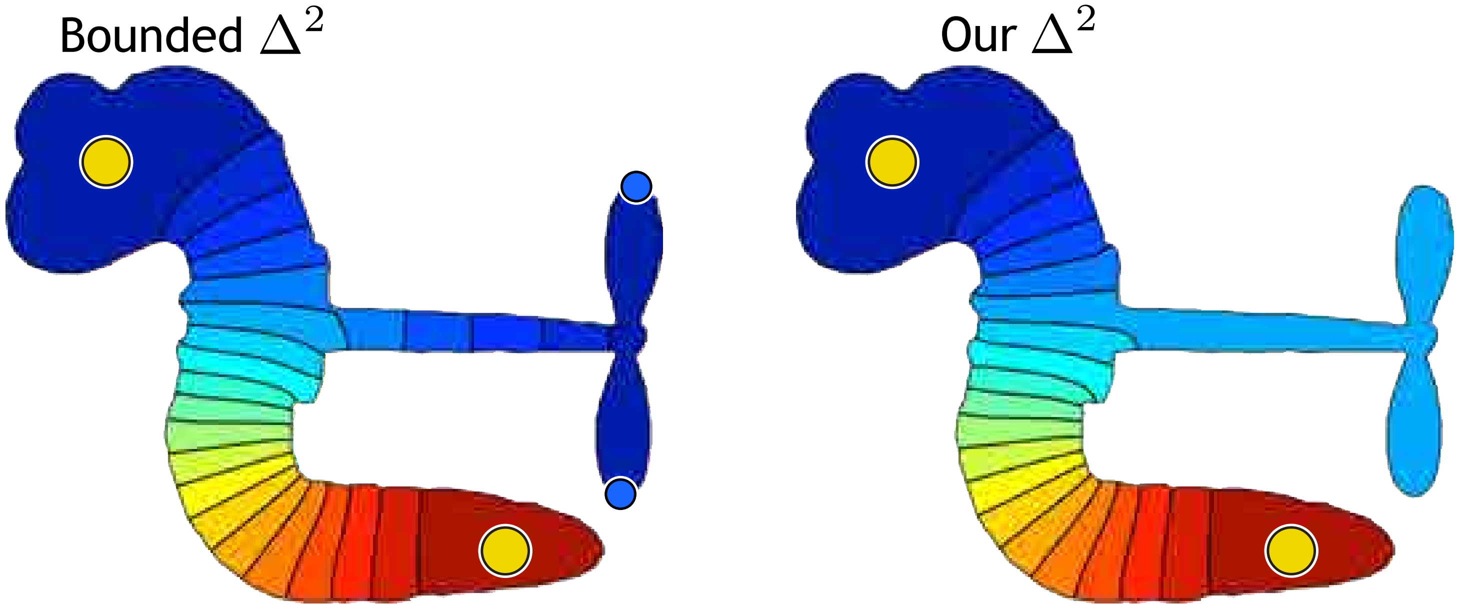

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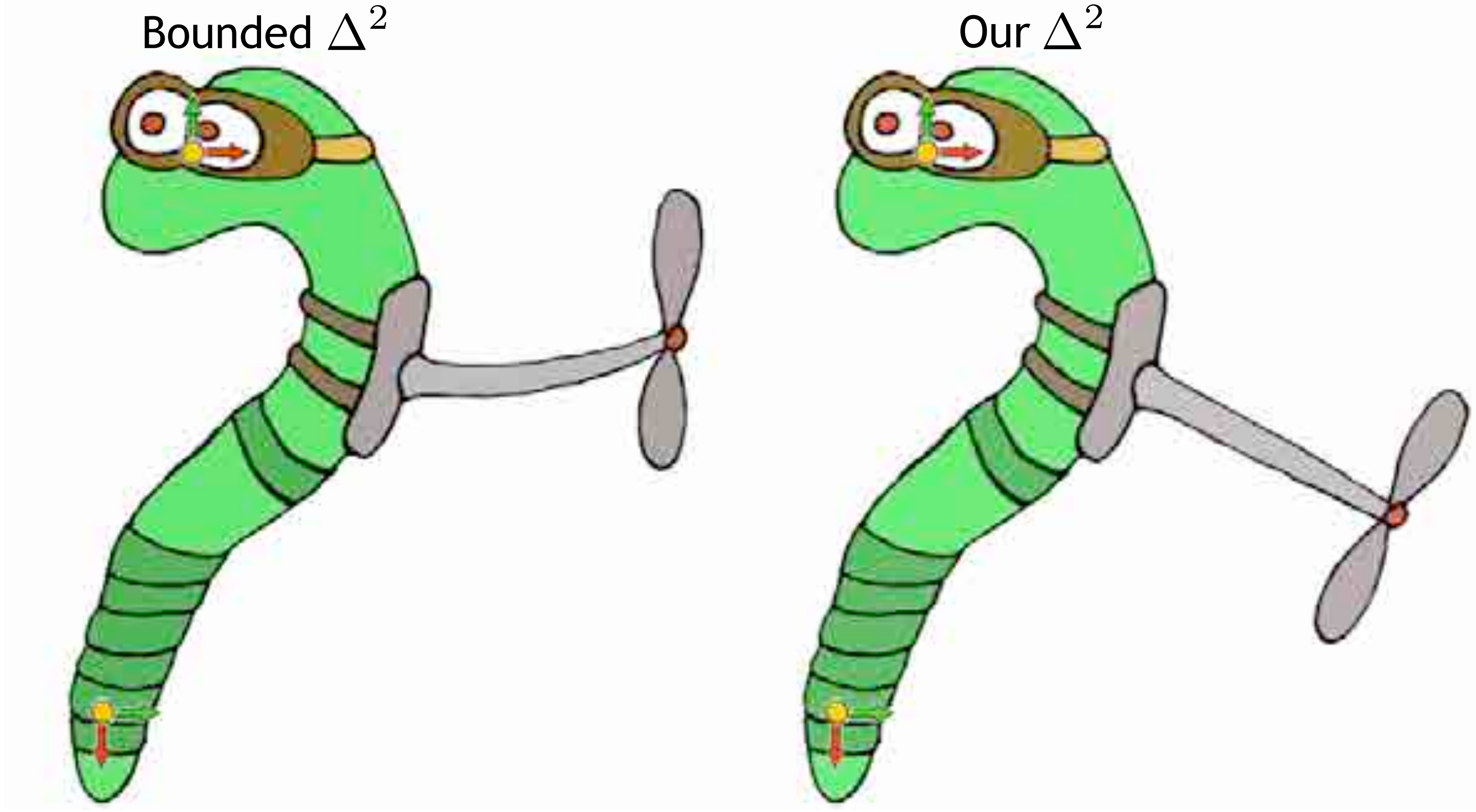
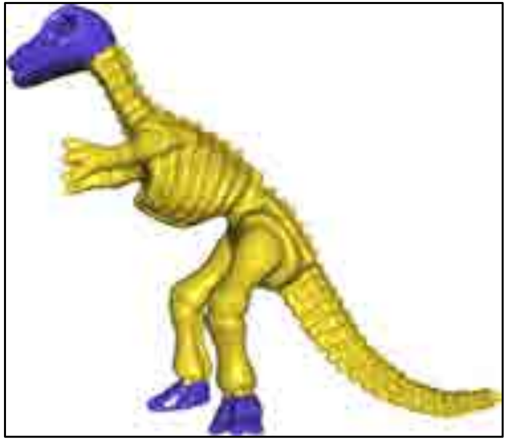
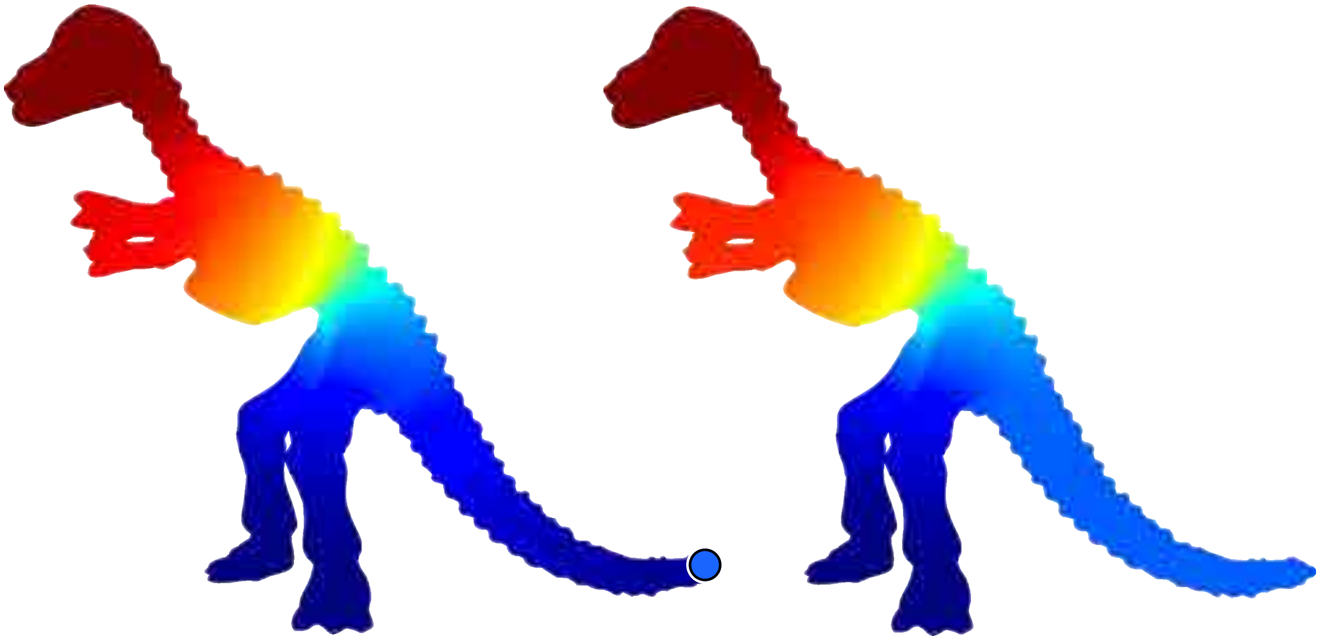


Our weights attach appendages to body

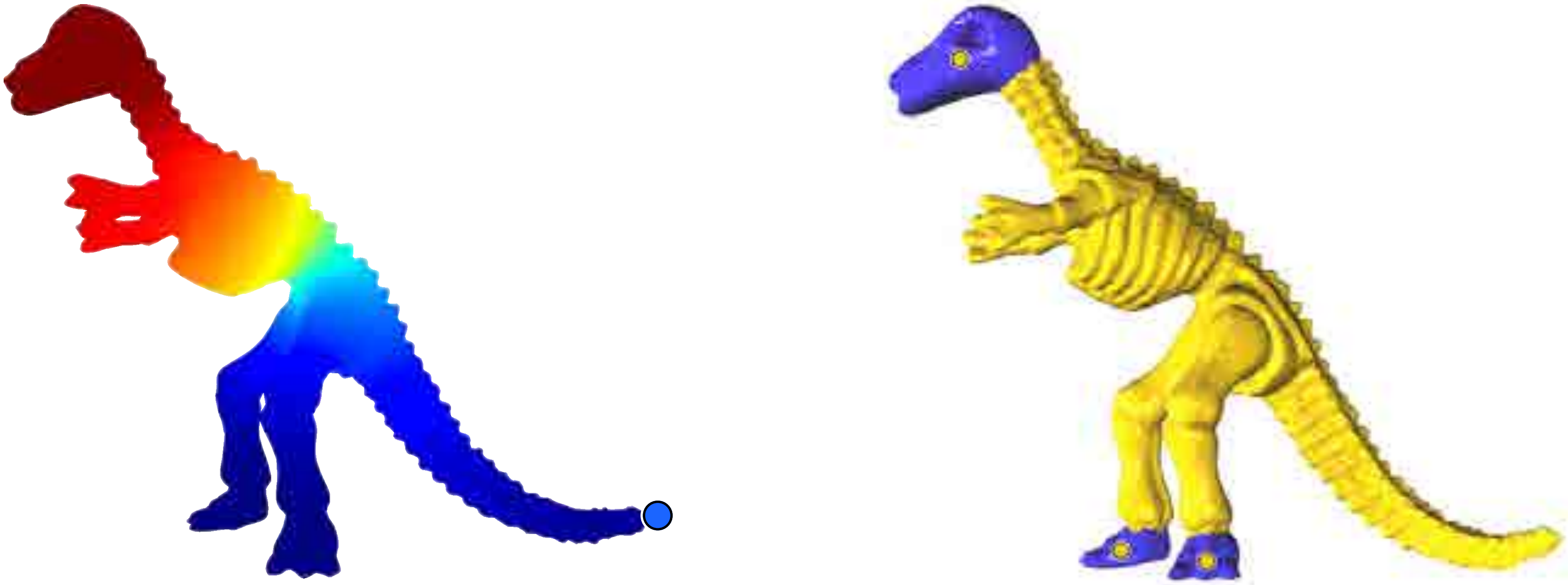




[Botsch & Kobbelt 2004, Jacobson et al. 2011]

Our method

Extrema glue appendages to far-away handles



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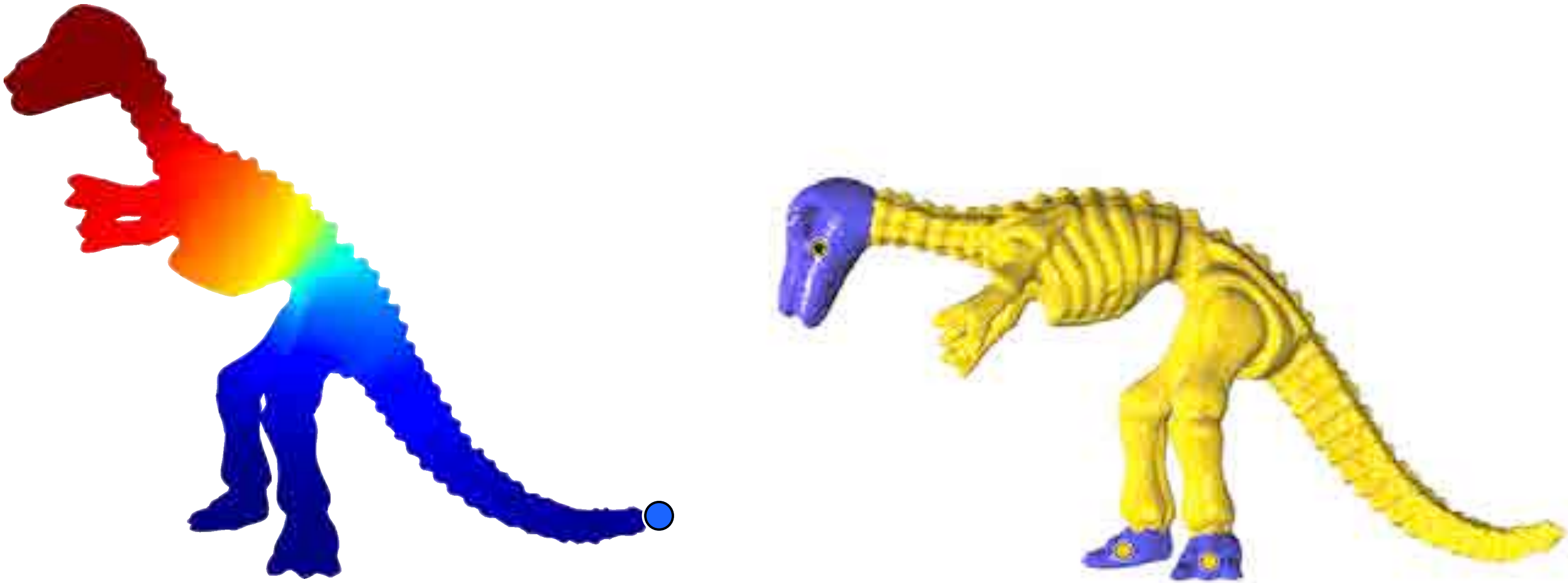

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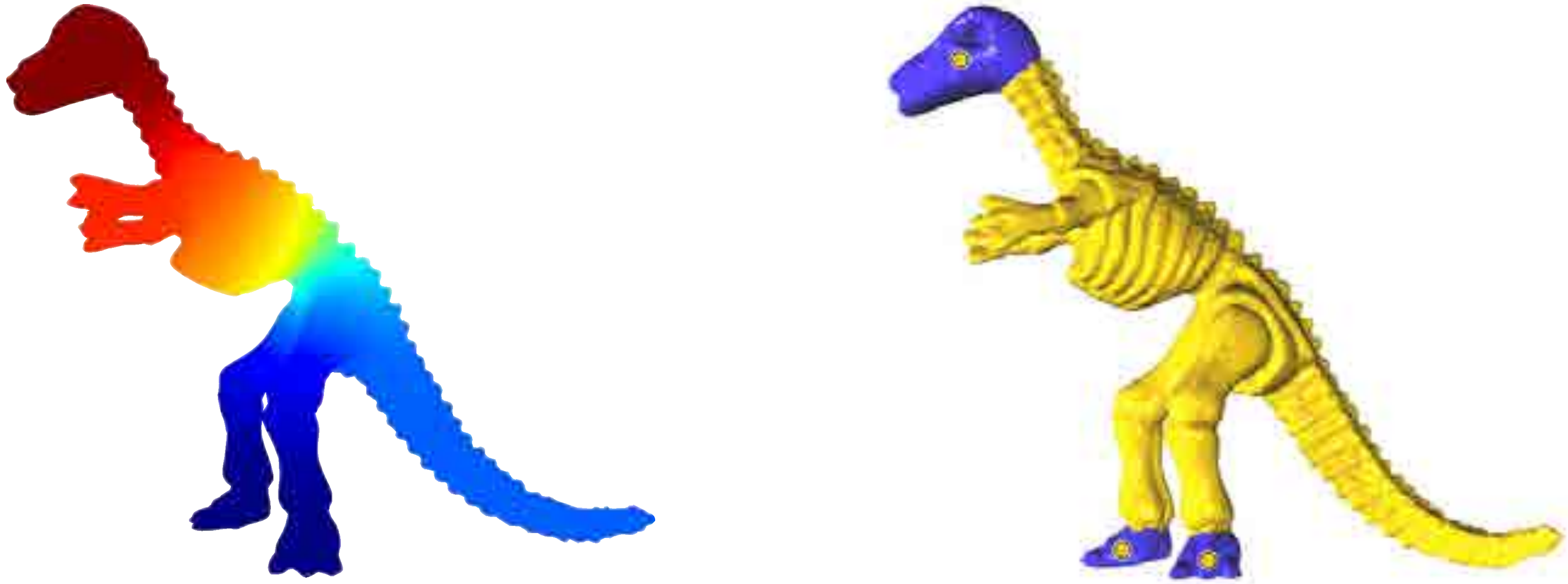

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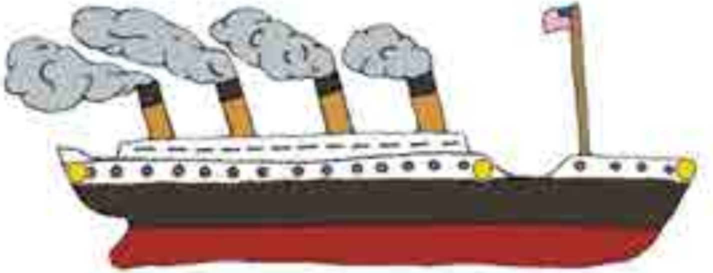
Extrema distort small features

Unconstrained Δ^2 [Botsch & Kobbelt 2004]

weight of middle point



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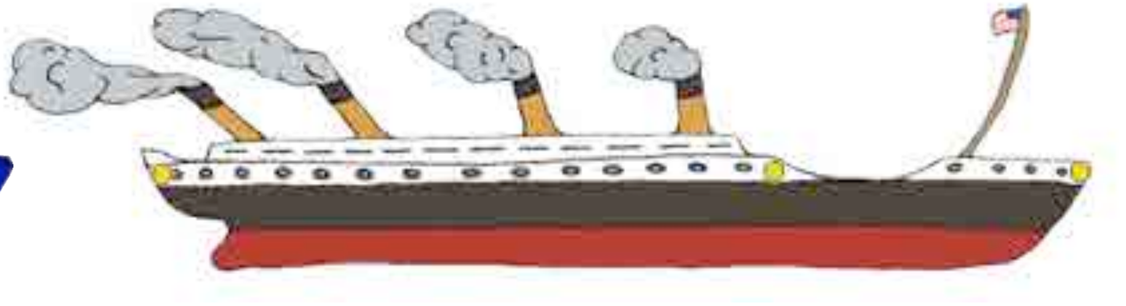

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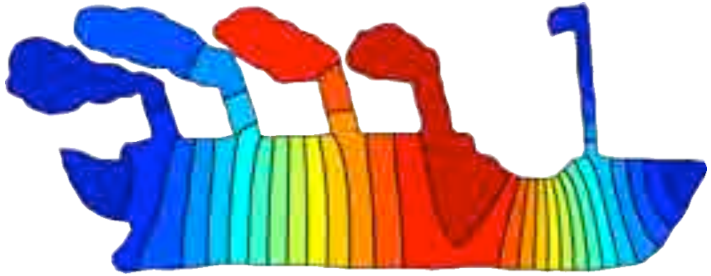

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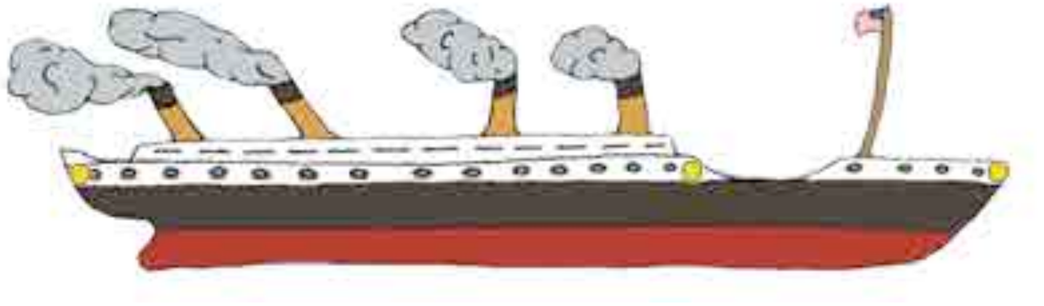
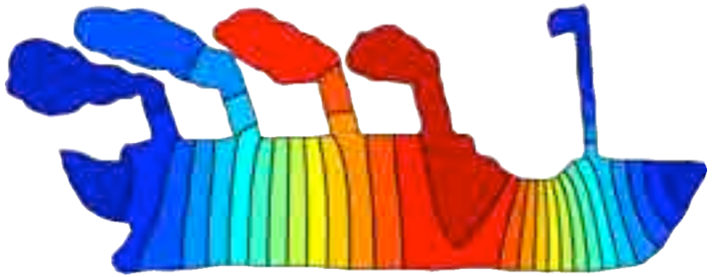
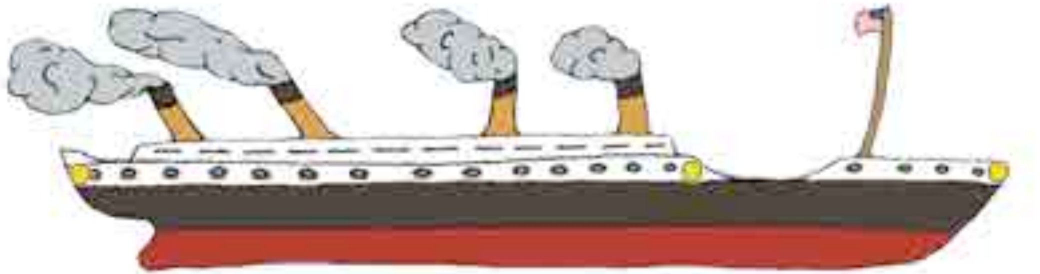


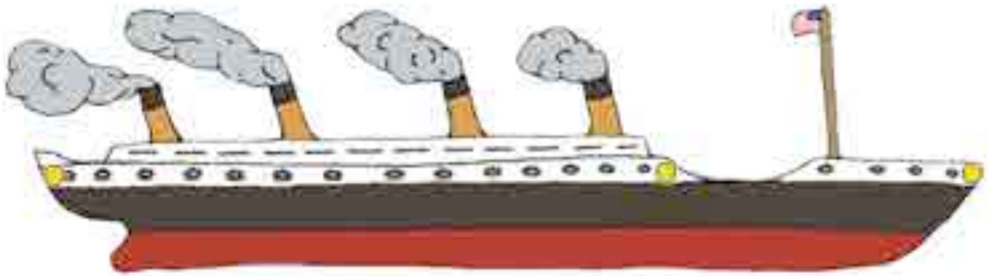
“Monotonicity” helps preserve small features

Bounded Δ^2 [Jacobson et al. 2011]

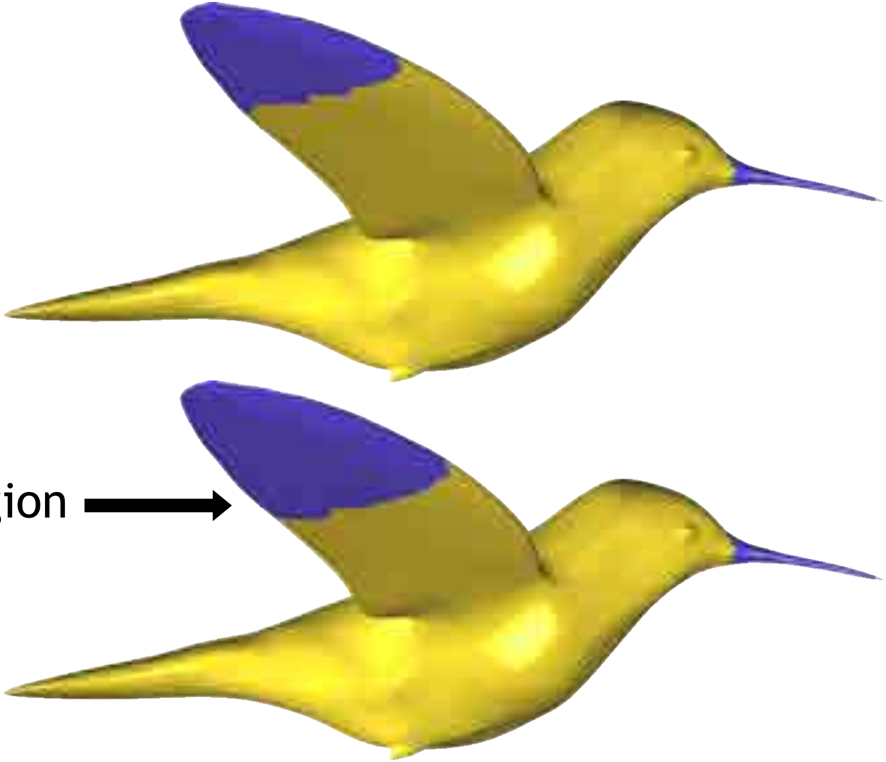




Our Δ^2



Spurious extrema are unstable, may “flip”



slightly larger region →

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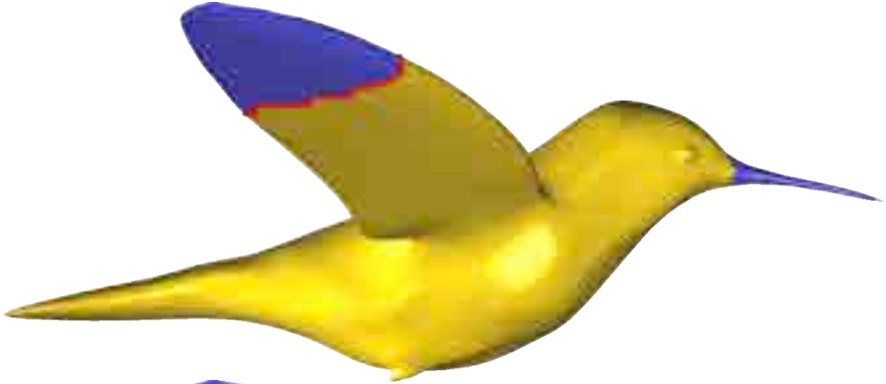

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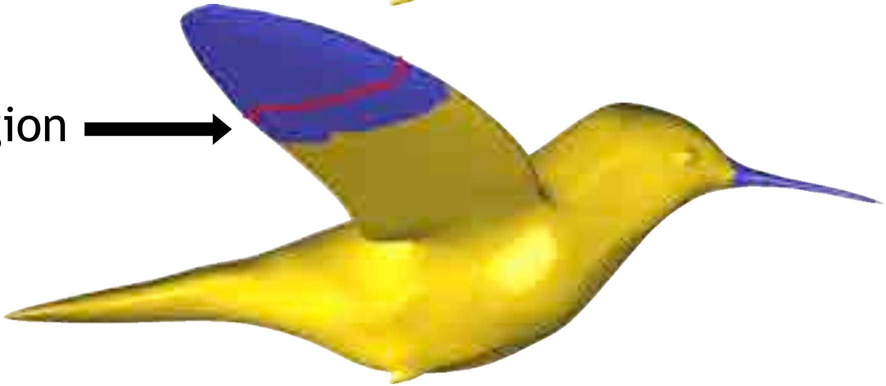

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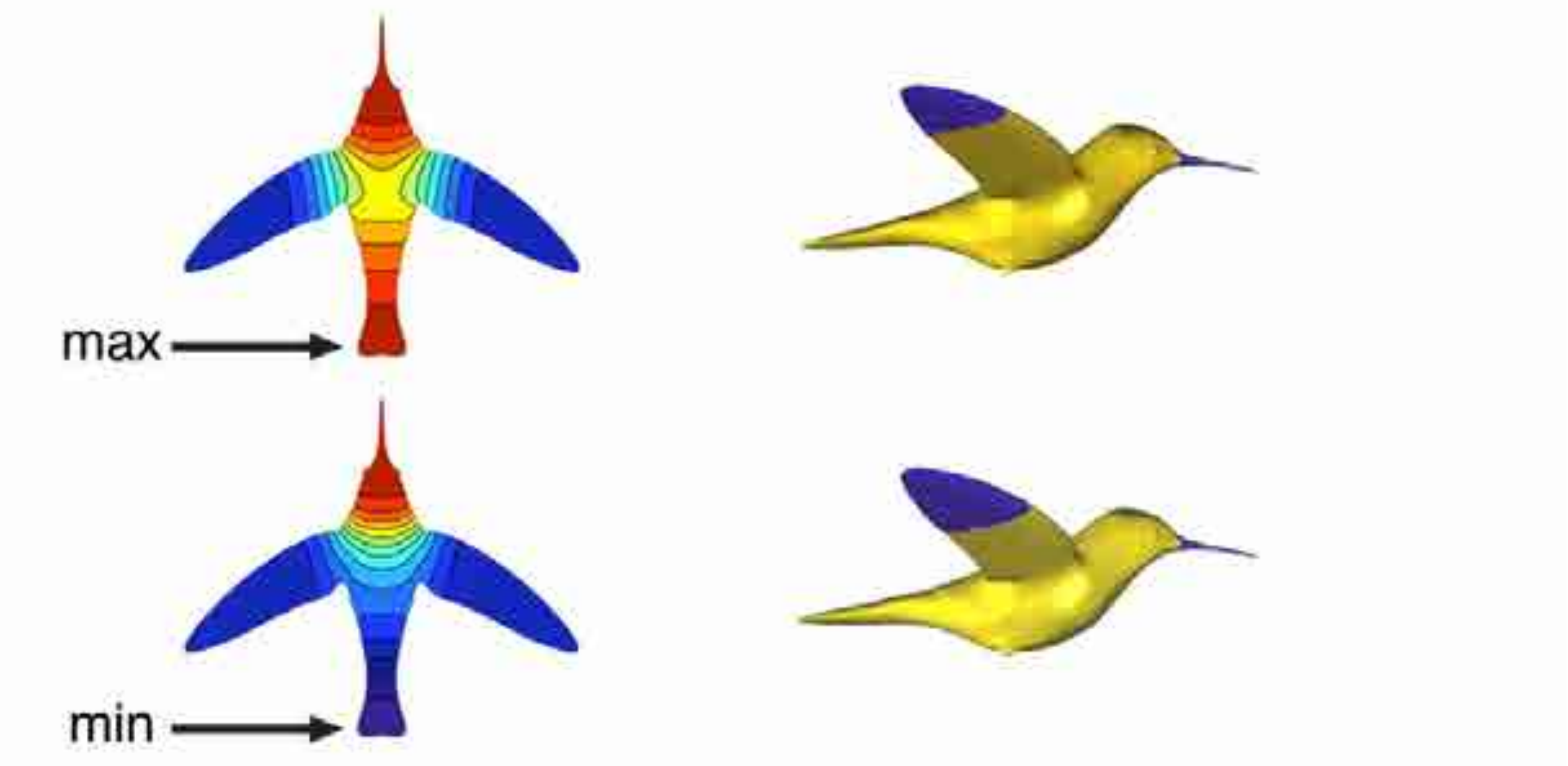

Unconstrained Δ^3 [Botsch & Kobbelt, 2004]

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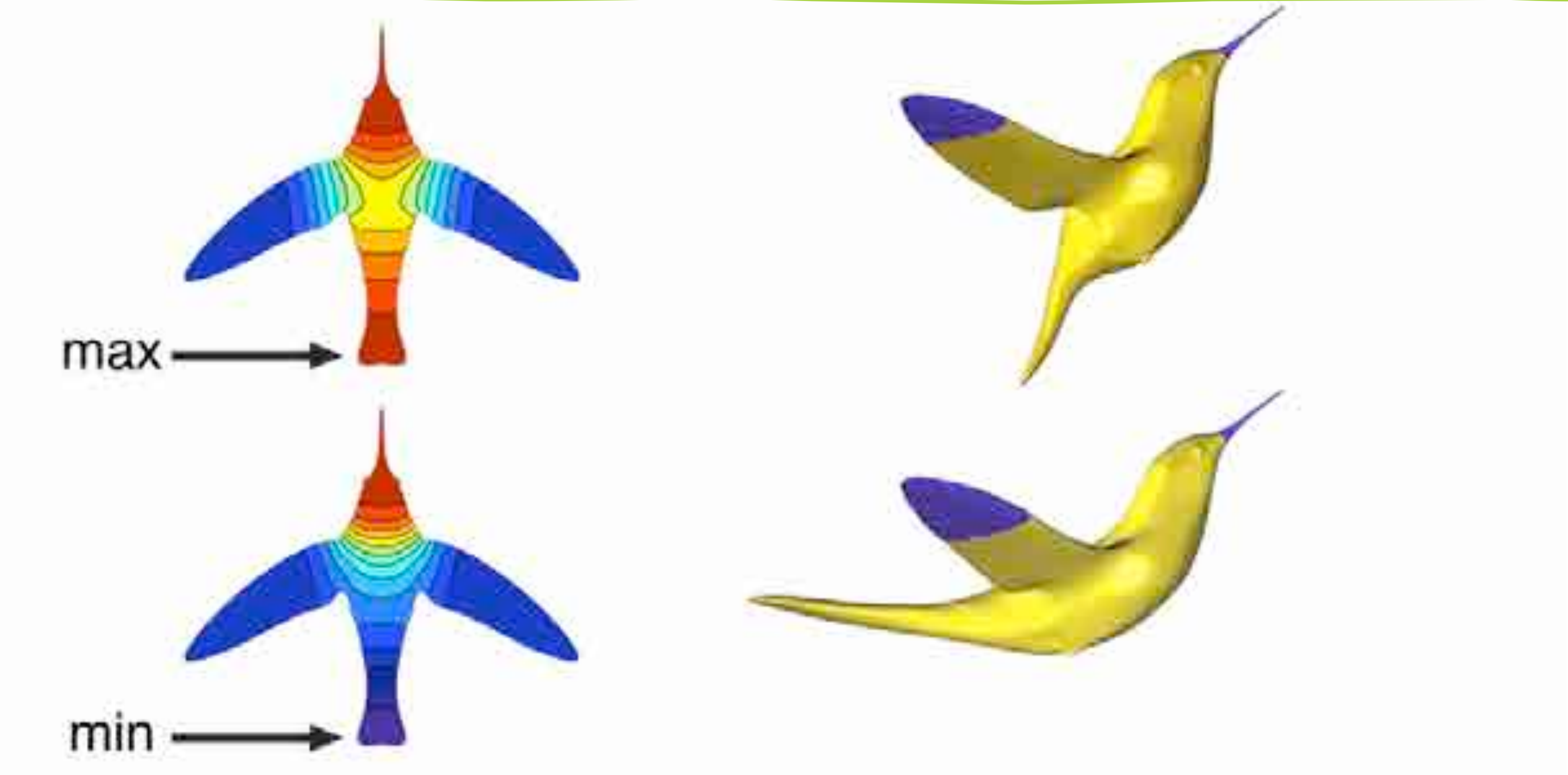

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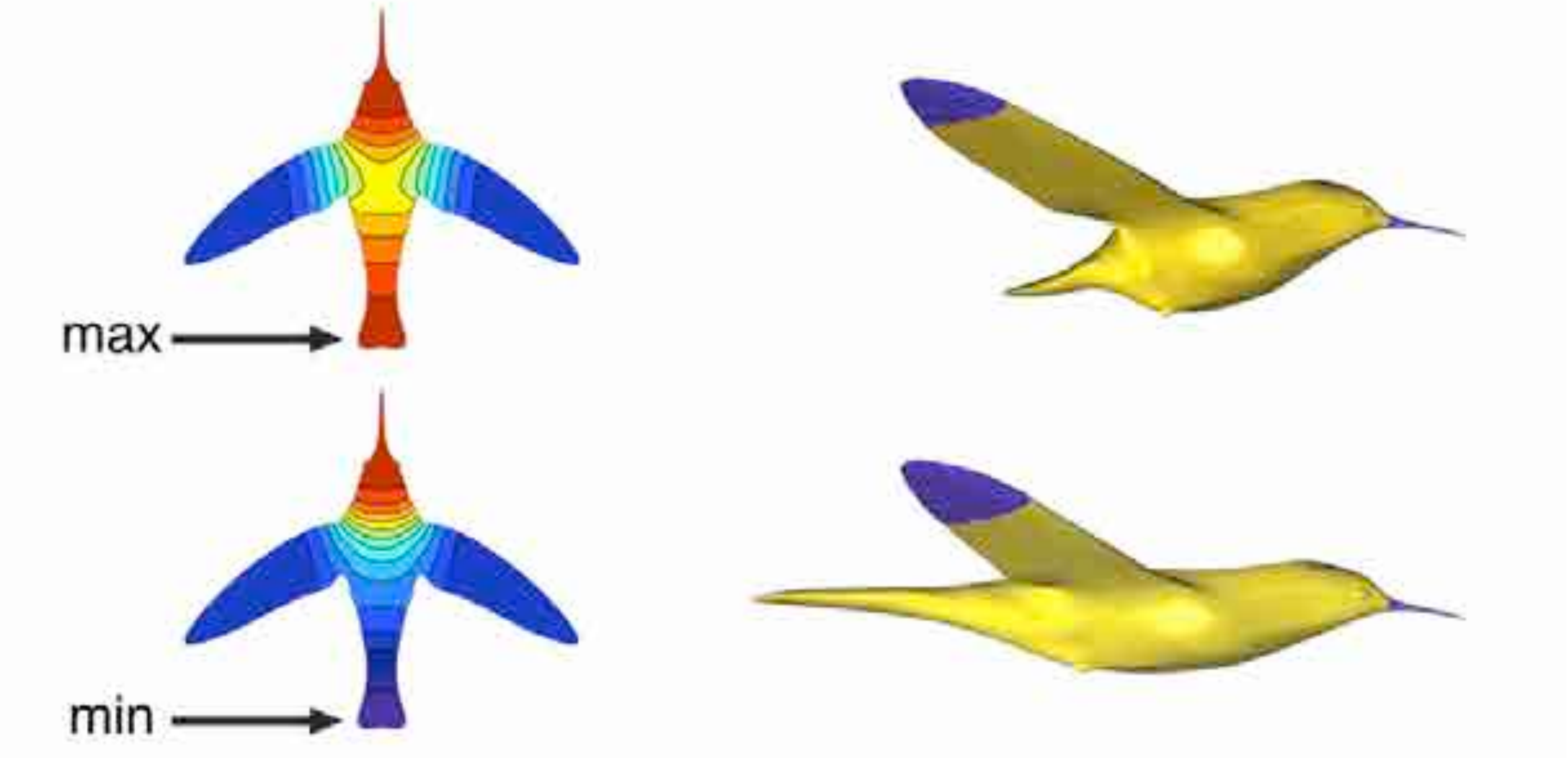

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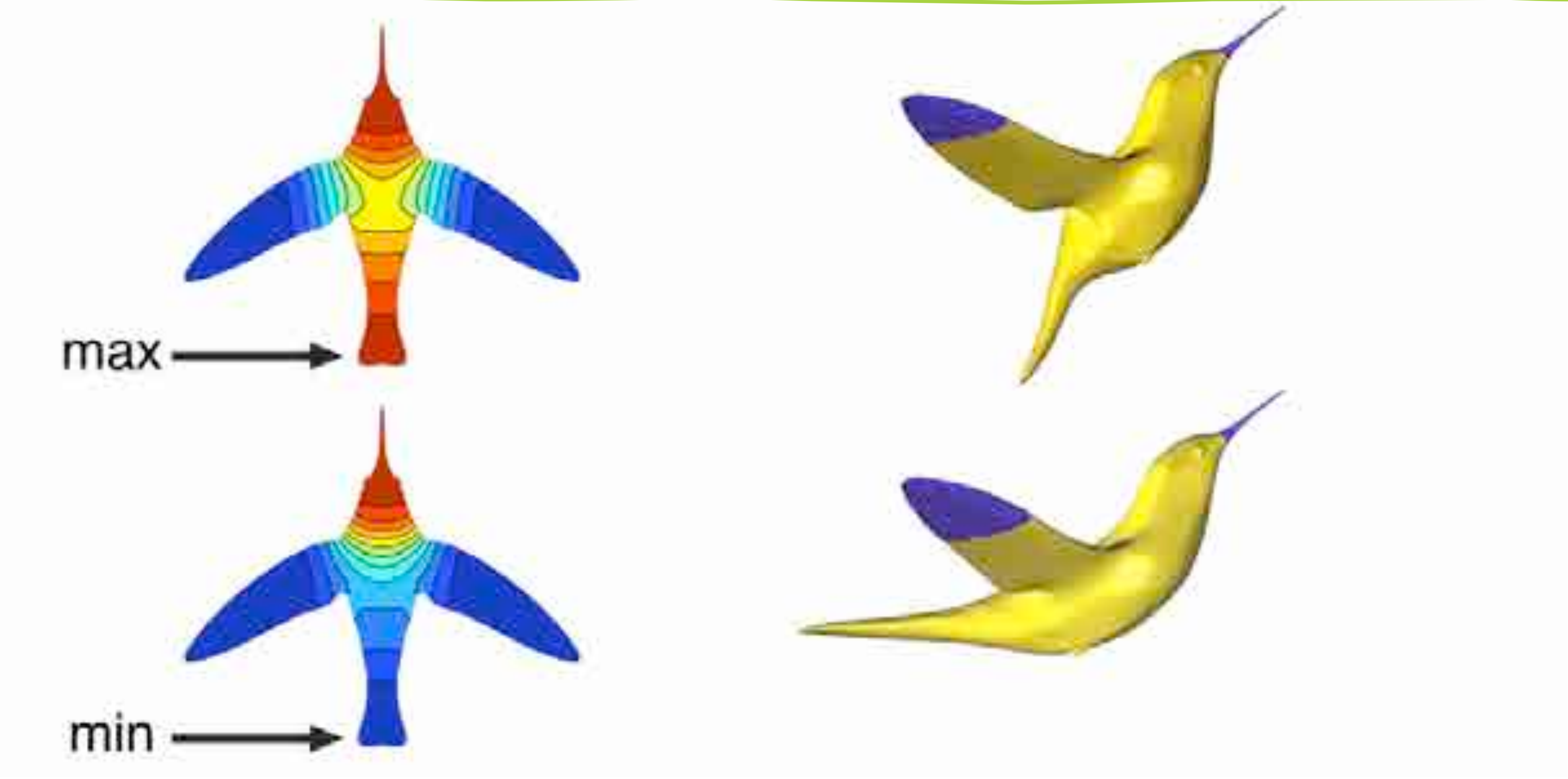

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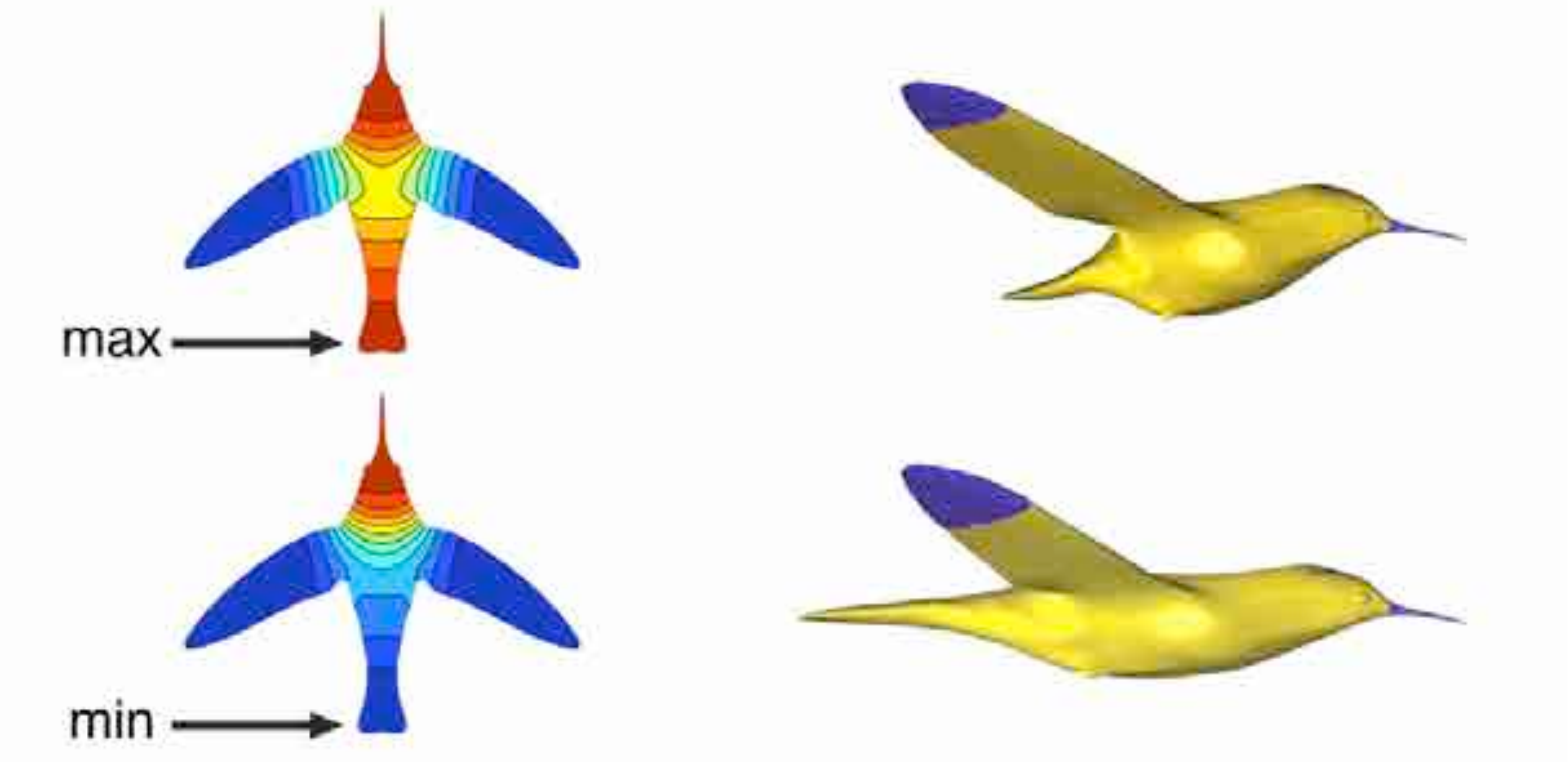
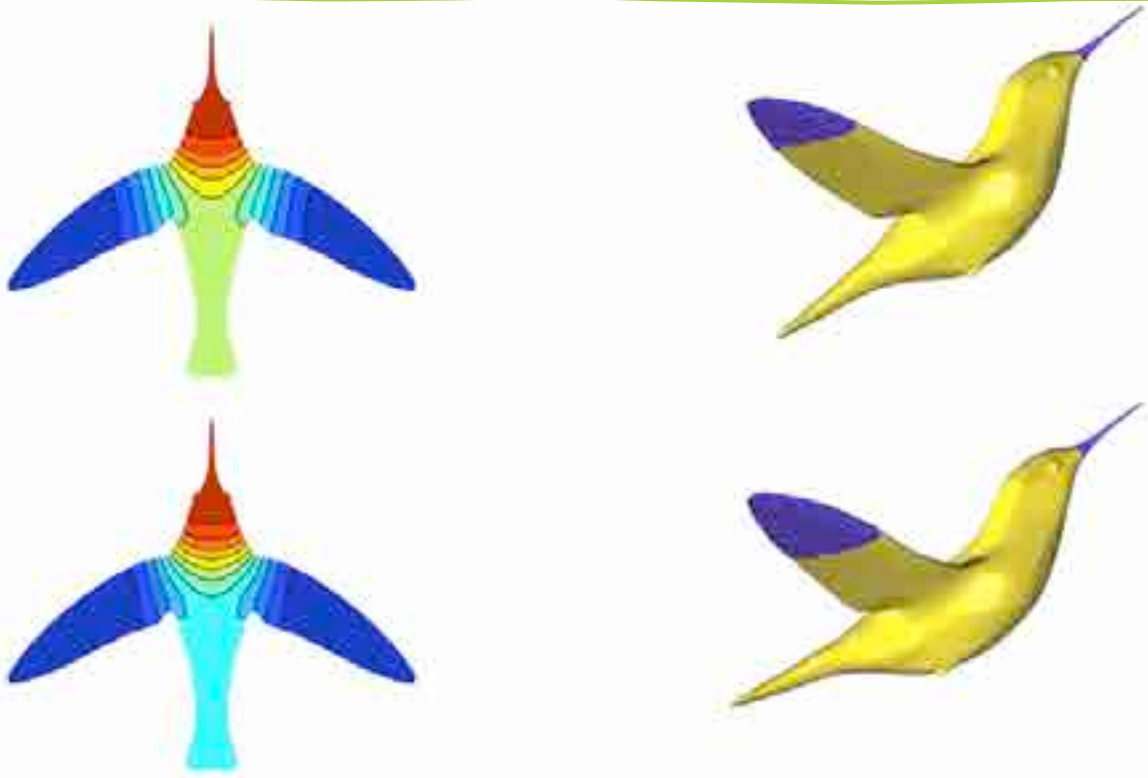


Bounded Δ^3

Lack of extrema leads to more stability



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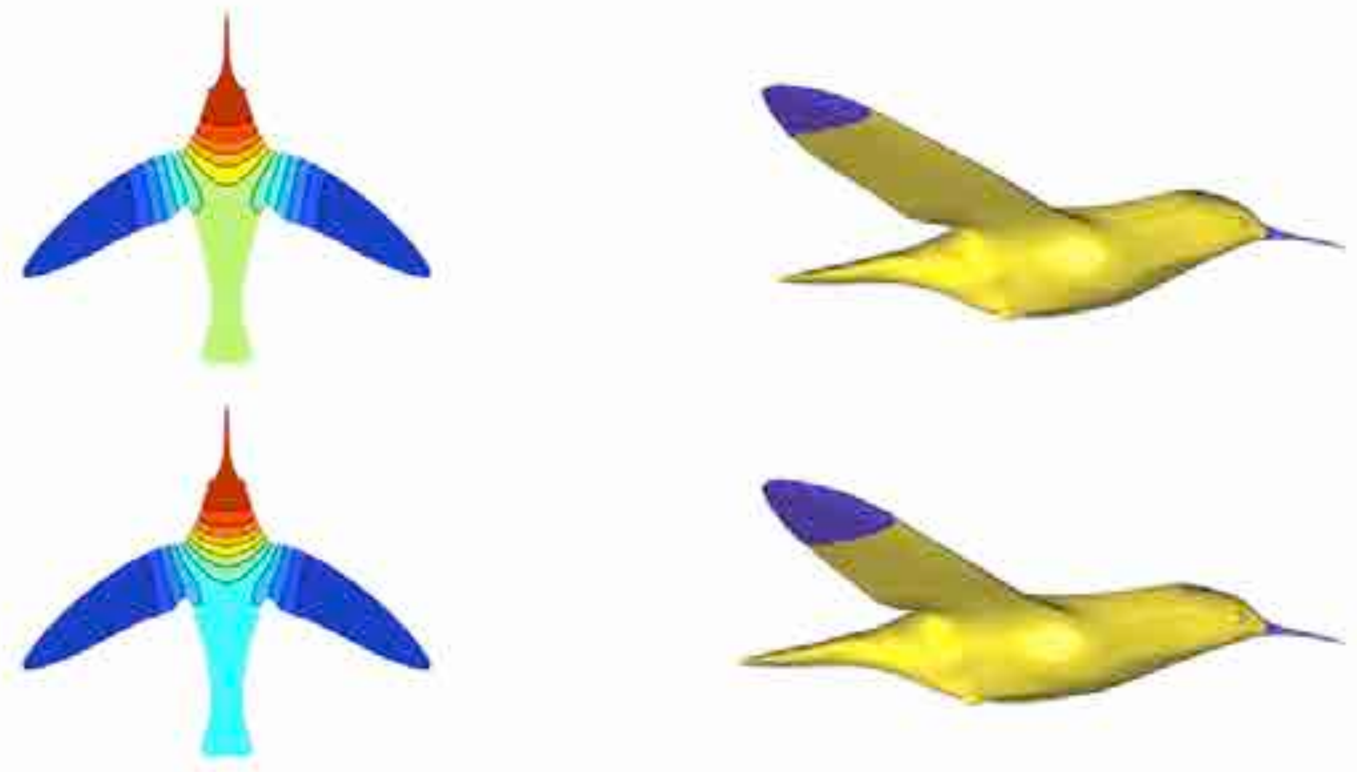
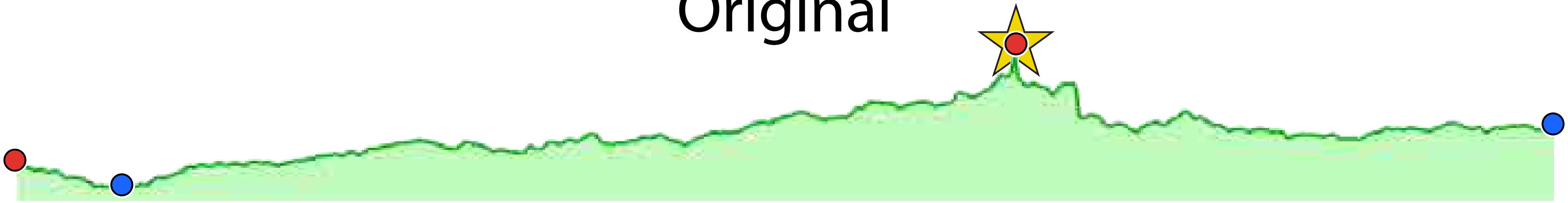


Our Δ^3

Even control continuity at extrema

Original



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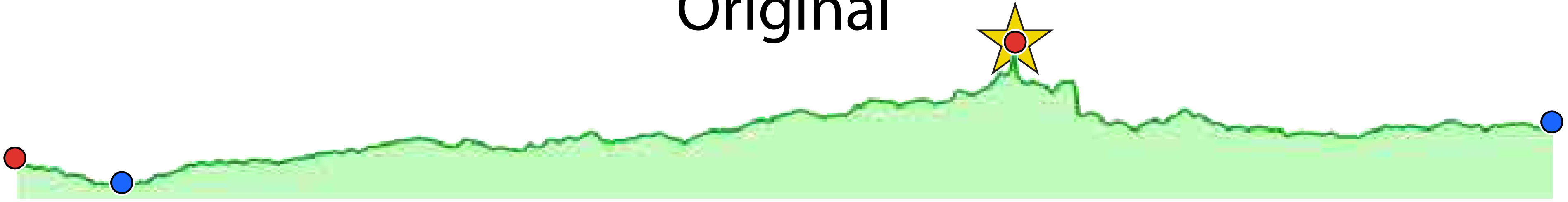
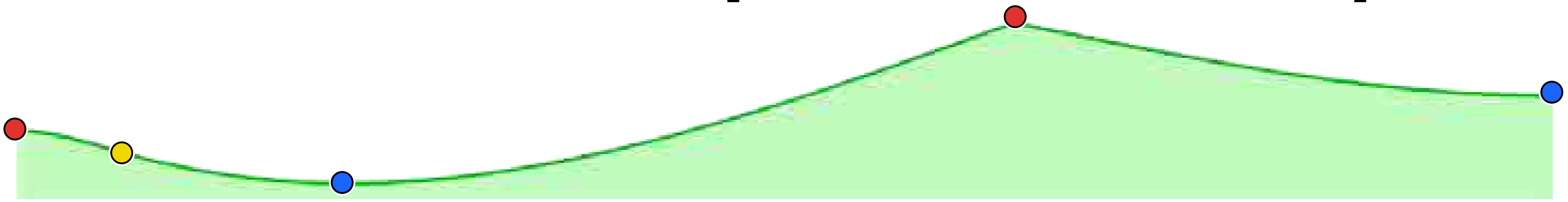


Direct extension of [Botsch & Kobbelt 2004]



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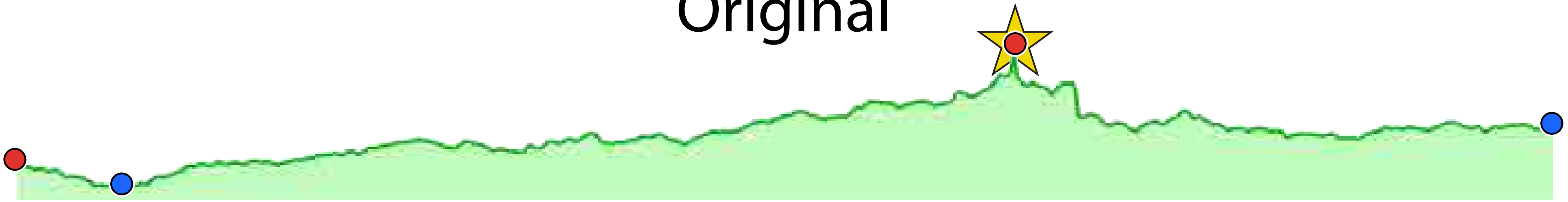
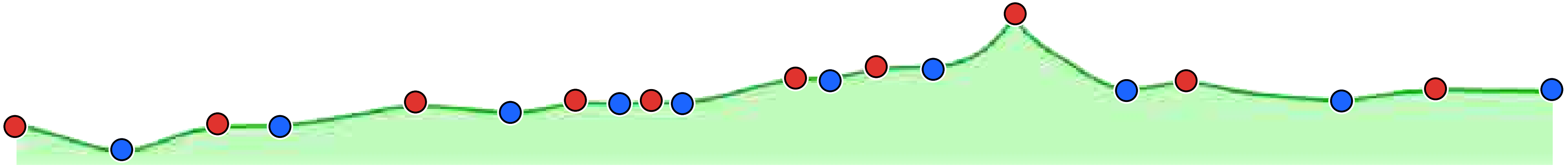


[Botsch & Kobbelt 2004] + data term



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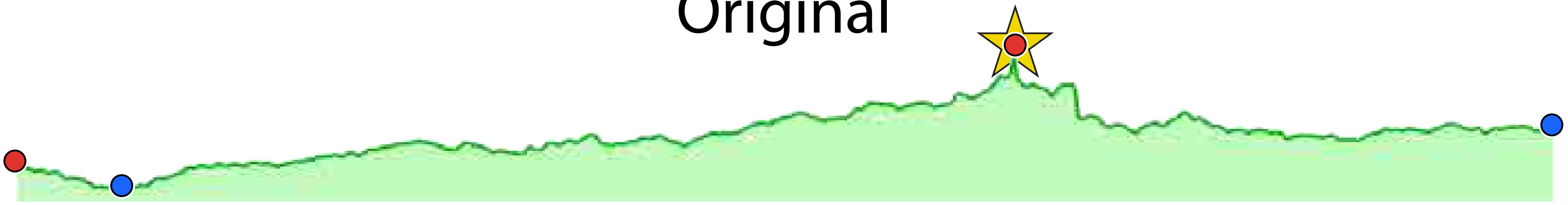
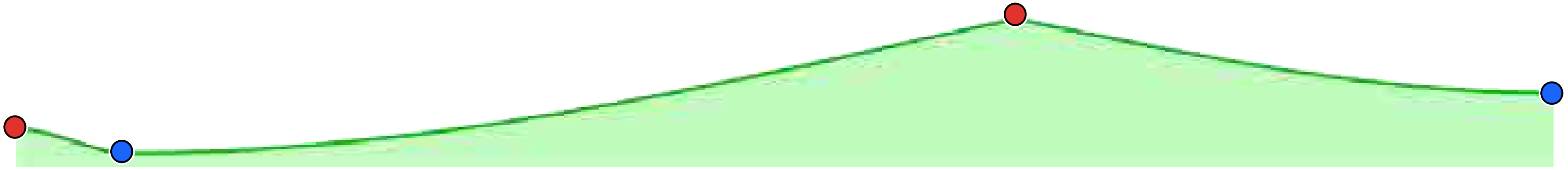


Our method without data term



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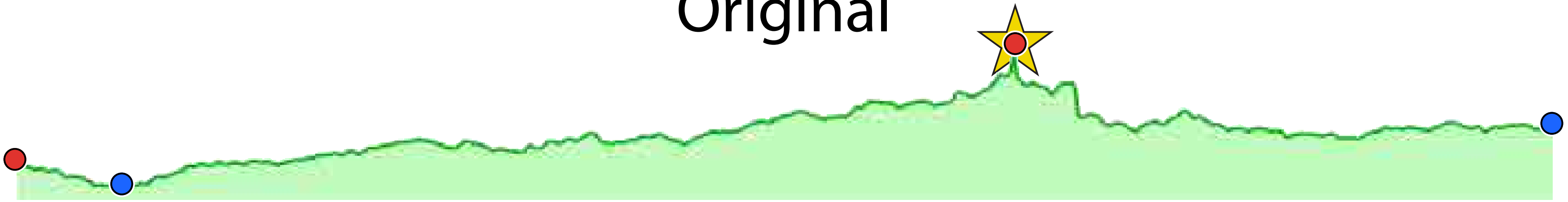

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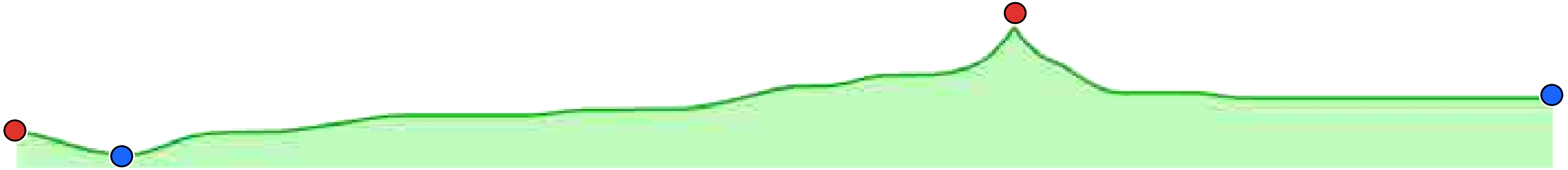
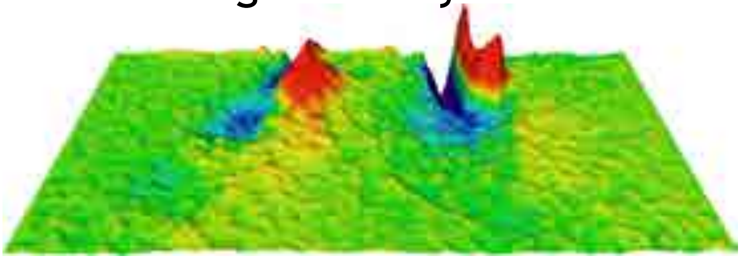


Reproduces results of Weinkauf et al. 2010…

Original noisy data



Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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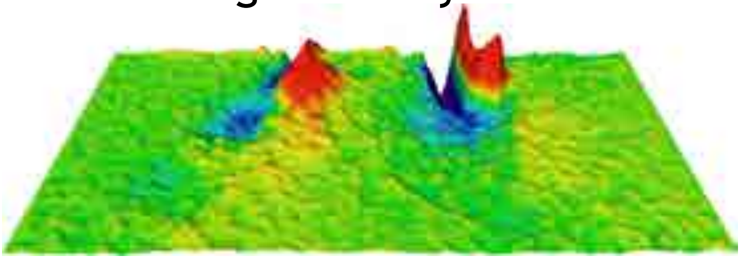

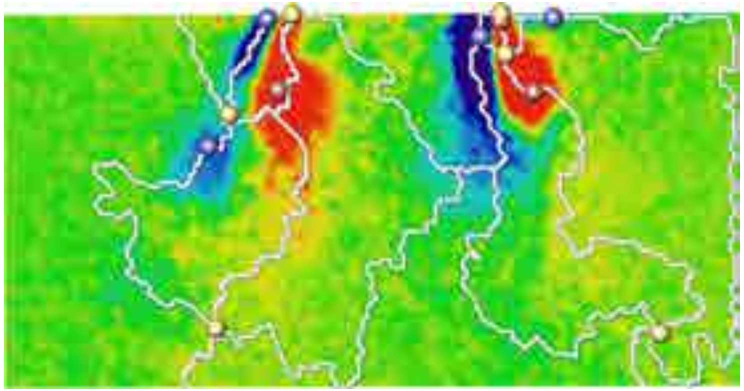

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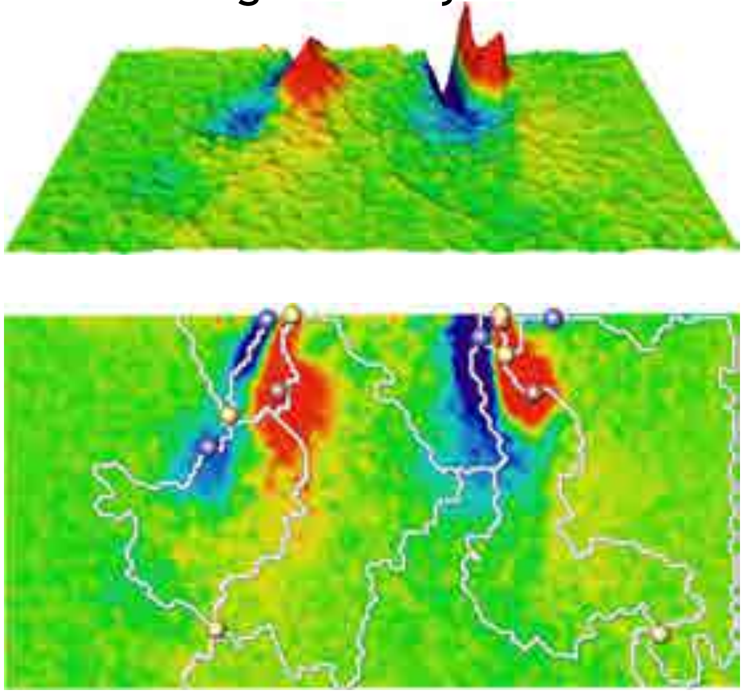
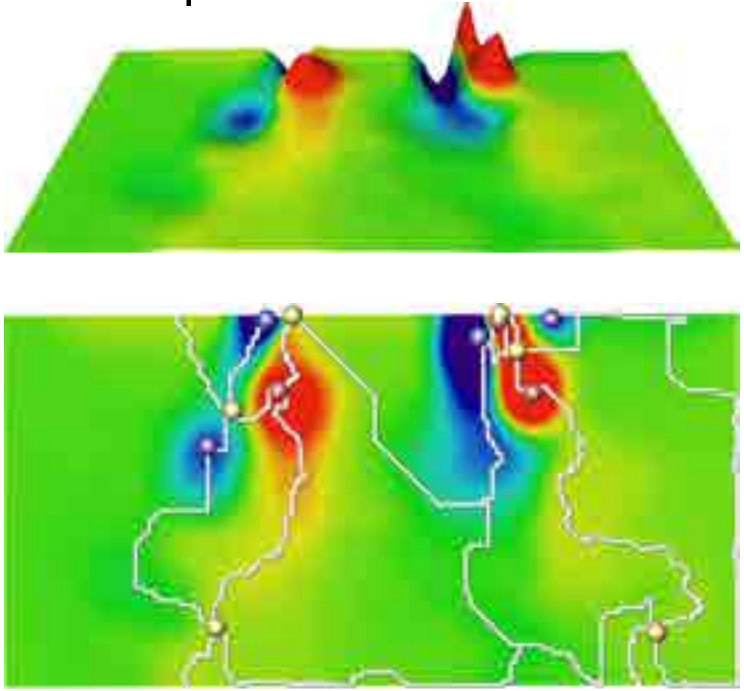


Simplified and smoothed



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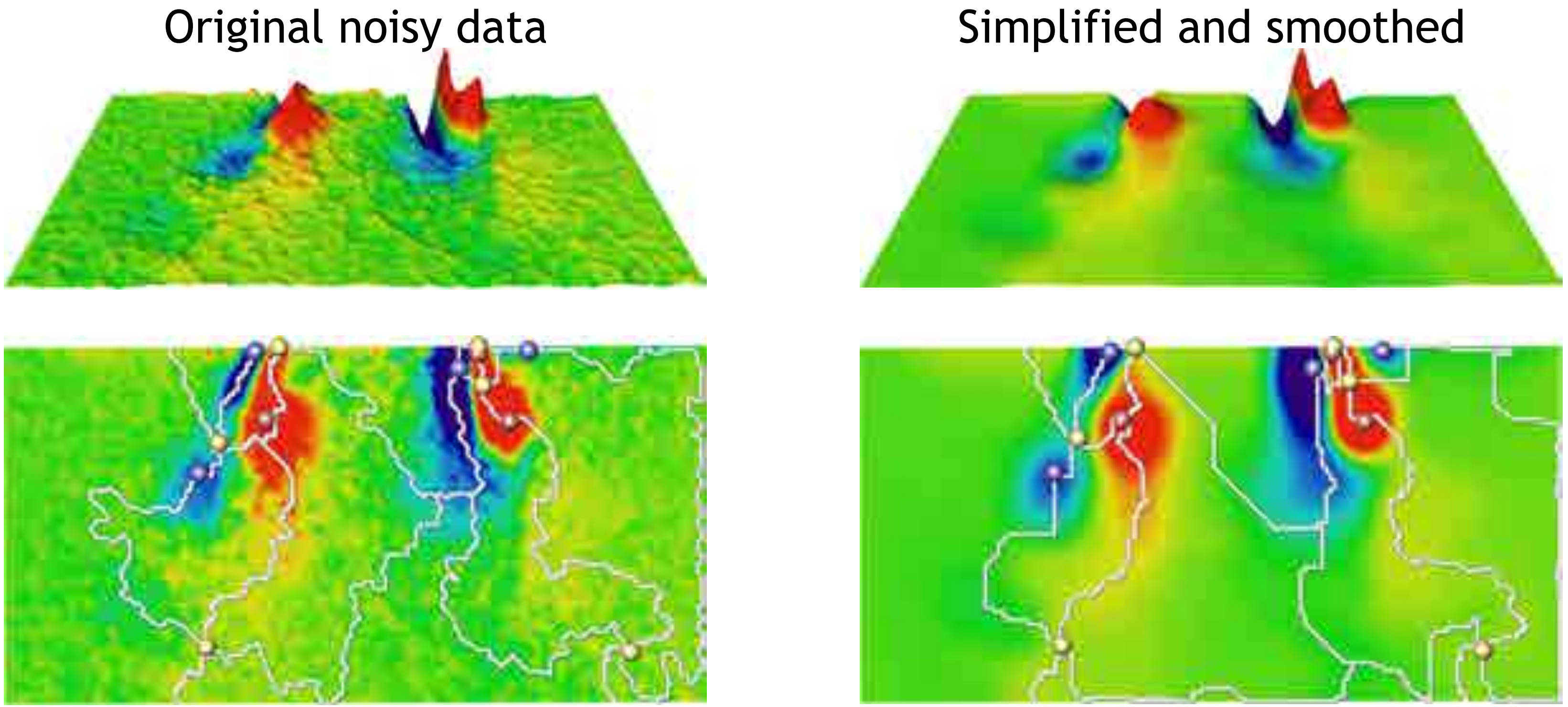
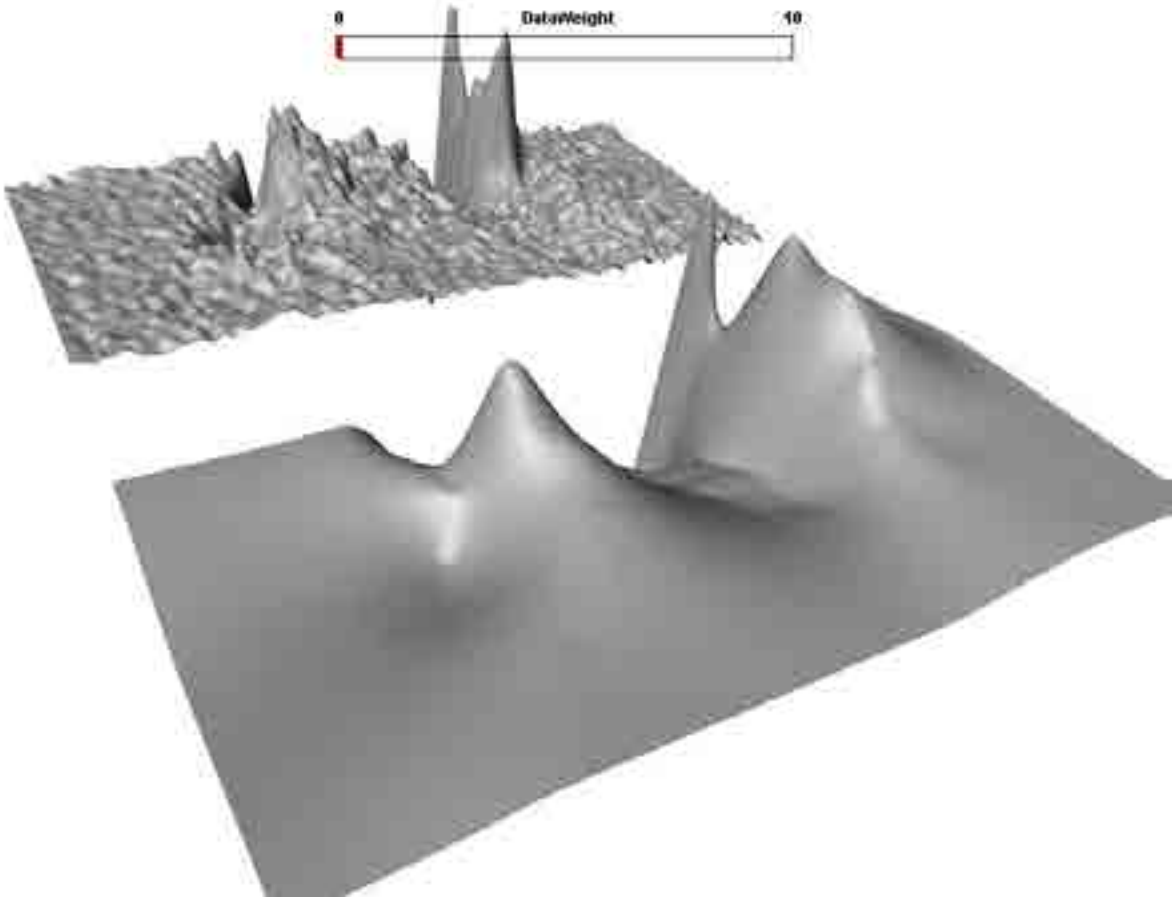


… but 1000 times faster



30K vertices
5 seconds per solve

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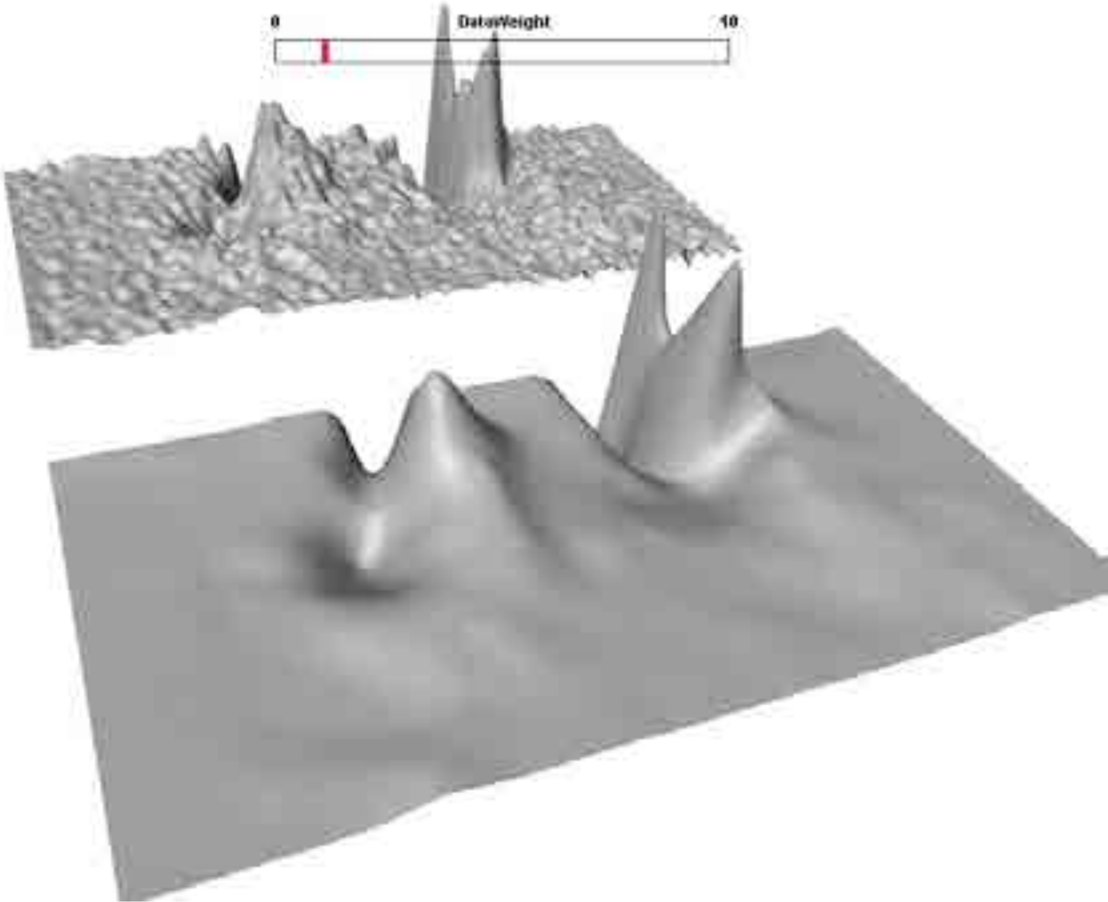

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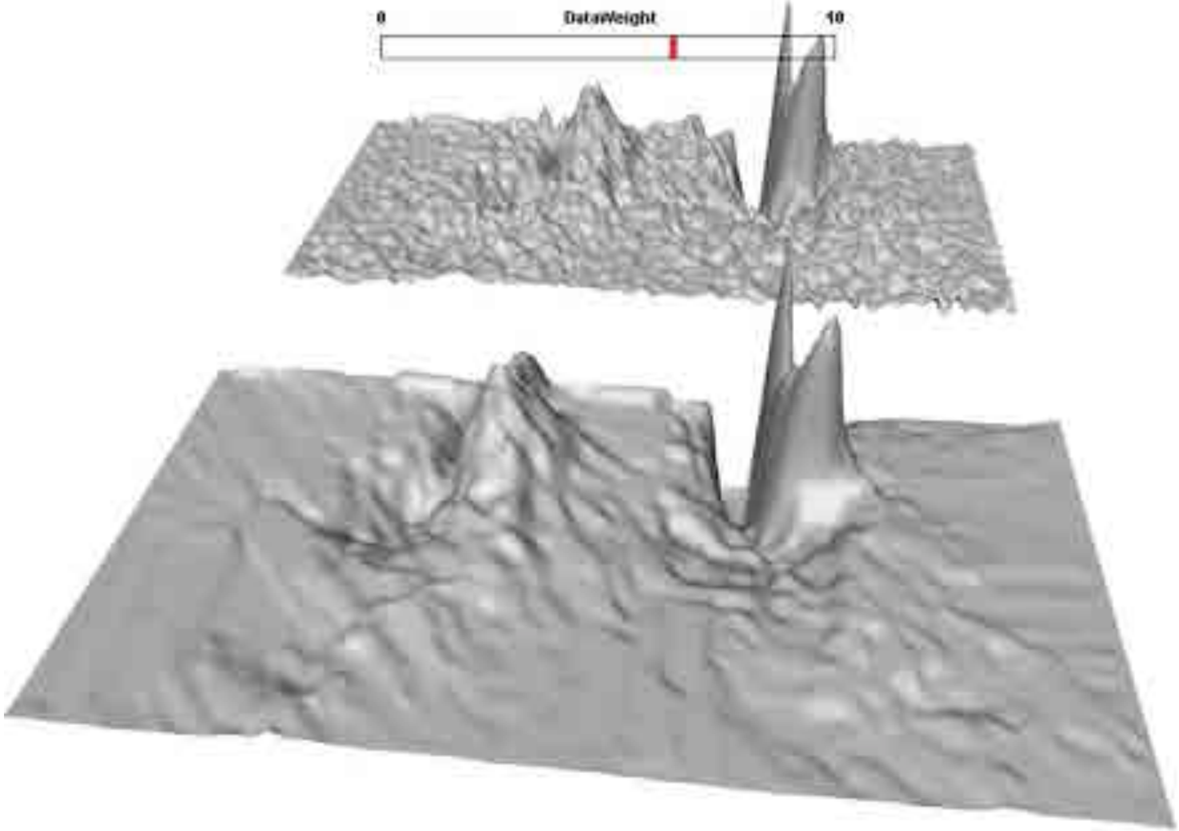

Conclusion: Important to control extrema

- Copy “monotonicity” of harmonic functions
- *Reduces* search-space, but optimization is tractable

Future work and discussion

- Larger, but still tractable subspace?
  - Consider all valid harmonic functions?

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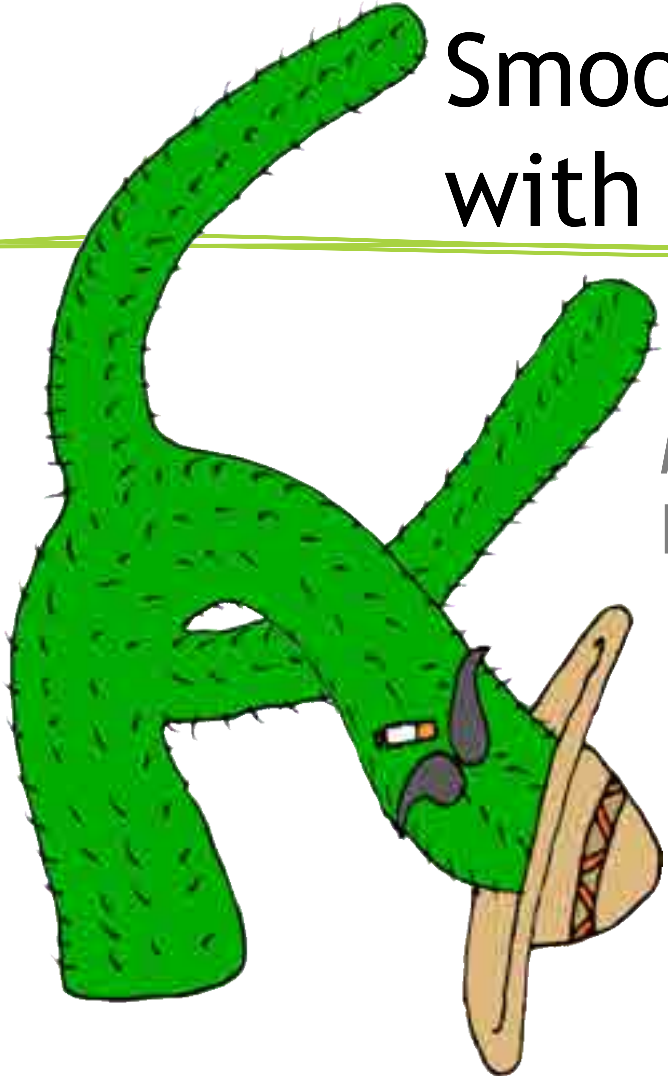
  - Consider all valid harmonic functions?
- Continuous formulation?

Acknowledgements

We thank Kenshi Takayama for his valuable feedback. This work was supported in part by an SNF award 200021_137879 and by a gift from Adobe Systems.

Smooth Shape-Aware Functions with Controlled Extrema



MATLAB Demo:
http://igl.ethz.ch/projects/monotonic/

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Tino Weinkauf
Olga Sorkine