# Phong Projection in Higher Dimensions

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**Abstract**

We supplement our paper *Weighted Averages on Surfaces* [1] with technical details, additional intuition, and proofs related to Phong projection. We also show how our generalized barycentric coordinates reduce to Moving Least Squares over Euclidean space. The motivation and applications of Phong projection are discussed in the paper [1].

## 1 Phong projection

Let $\mathcal{M} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a triangle mesh with vertices $\mathcal{V} \subset \mathbb{R}^D$. Each vertex has an associated tangent plane (taken e.g. from the Loop limit surface), represented by two basis vectors, which we assume to be orthonormal. Consider a triangle with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and denote the tangent planes at these vertices as $T_1, T_2, T_3 \in \mathbb{R}^{2\times D}$. Let $\Psi(\xi_1, \xi_2, \xi_3)$ be an interpolated basis for the tangent plane at the point on the triangle with barycentric coordinates $\xi_1, \xi_2, \xi_3$.

The function $\Psi : \mathbb{R}^3 \to \mathbb{R}^{2\times D}$ must continuously interpolate the tangent planes to the triangle interiors over the entire mesh. Defining $\Psi$ is a non-trivial task (which we will tackle later) because a tangent plane can be specified using different bases, while the interpolant should be independent of this choice of basis and also consistent on edges and vertices shared by multiple triangles. Once we do have such a $\Psi$, we can define Phong projection, as in the paper:

**Definition 1.1.** *A point $\hat{\mathbf{p}} = \xi_1\mathbf{v}_1 + \xi_2\mathbf{v}_2 + \xi_3\mathbf{v}_3$ on triangle $\mathbf{t}$ with vertices $\mathbf{v}_i$ is a Phong projection of $\mathbf{p} \in \mathbb{R}^D$ if:*

$$\Psi(\xi_1, \xi_2, \xi_3)\,(\xi_1\mathbf{v}_1 + \xi_2\mathbf{v}_2 + \xi_3\mathbf{v}_3 - \mathbf{p}) = \mathbf{0}, \tag{1}$$

$$\xi_1 + \xi_2 + \xi_3 = 1, \tag{2}$$

$$\xi_i \geq 0. \tag{3}$$

**Definition 1.2.** *The Phong projection of a point $\mathbf{p}$ onto a triangle mesh $\mathcal{M}$ is the closest Phong projection with respect to every triangle of $\mathcal{M}$.*

Note that the Phong projection onto even a single triangle is generally not unique. Consider the affine subspace through $\mathbf{v}_1$ orthogonal to $T_1$. All points in that subspace project to $\mathbf{v}_1$. If the intersection between this subspace and the analogous one for $\mathbf{v}_2$ is not empty (as will generally happen for $D \geq 4$), both $\mathbf{v}_1$ and $\mathbf{v}_2$ will be Phong projections of points in the intersection. In the paper, we provide experimental evidence that for reasonable meshes and points $\mathbf{p}$ close to the mesh, this does not happen; when it does, we break ties arbitrarily. Also, unlike Euclidean projection, Phong projection might not exist at all. In Section 3 of this document, we give an outline of how one might prove that for well-tessellated meshes Phong projection is guaranteed to exist for points $\mathbf{p}$ close to the mesh.

The remainder of this document is organized as follows. In Sections 1.1-1.4 we deal with tangent plane interpolation and define $\Psi$, first for mesh edges and then for triangle interiors. Section 2 shows that $\Psi$ is continuous under some mild conditions. In Section 3 we give an informal sketch of how to prove that the Phong projection based on $\Psi$ is well-defined. Throughout these sections, we use some simple algebraic results;

we concentrate all these auxiliary propositions and their proofs in Section 4 in order to avoid clutter in the exposition. Finally, in Section 5 we show the equivalence between our generalized barycentric coordinates (see Section 3.4 in the paper) and Moving Least Squares when working in Euclidean spaces.

## 1.1 Plane representation

We now discuss how to work with planes in a $D$-dimensional space, each represented by a basis encoded in a 2-by-$D$ matrix.

**Definition 1.3.** *Let $T, K \in \mathbb{R}^{2\times D}$ be two rank-2 matrices. If there exists a non-singular matrix $A \in \mathbb{R}^{2\times 2}$ for which $K = AT$, then $T$ and $K$ represent the same plane. We call such pairs of 2-by-D matrices equivalent and write $T \equiv K$.*

If $T$ and $K$ are equivalent and each has orthonormal rows, the matrix $A$ relating them is orthogonal.

**Definition 1.4.** *For $T \in \mathbb{R}^{2\times D}$, we denote by $\mathrm{Ort}(T)$ the nearest orthonormal basis to $T$, i.e.,*

$$\mathrm{Ort}(T) = \operatorname*{argmin}_{B\in\mathbb{R}^{2\times D}\,:\,BB^T=I} \|T - B\|.$$

In the definition above and for the remainder of this document, the matrix norm $\|\cdot\|$ stands for the Frobenius norm, unless explicitly stated otherwise.

**Definition 1.5.** *If $T$ and $K$ both have rank 2, we measure the* distance *between the planes they represent as*

$$d(T, K) = \min_{A\in O(2)} \|\mathrm{Ort}(T) - A\,\mathrm{Ort}(K)\|.$$

Letting $T' = \mathrm{Ort}(T)$ and $K' = \mathrm{Ort}(K)$, we show in Proposition 4.4 that at the minimum, $A = \mathrm{Ort}(T'K'^T)$ and that this distance is equivalent up to a constant to the more standard projection operator distance $\|T'^T T' - K'^T K'\|$ (Propositions 4.6 and 4.7).

## 1.2 Interpolation requirements

The interpolation operator $\Psi$ needs to satisfy some simple conditions for the interpolation to work. Given barycentric weights $\Xi = (\xi_1, \xi_2, \xi_3)$, where $\xi_1 + \xi_2 + \xi_3 = 1,\ \xi_i \geq 0$, we want to find a blended plane $\Psi(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^{2\times D}$ such that:

**Interpolation at vertices.** $\Psi(1, 0, 0) \equiv T_1,\ \Psi(0, 1, 0) \equiv T_2,\ \Psi(0, 0, 1) \equiv T_3$.

**Interpolation at edges.** For $\xi_1, \xi_2 \geq 0,\ \xi_3 = 0$, $\Psi(\xi_1, \xi_2, \xi_3)$ does not depend on $T_3$ or $\mathbf{v}_3$ (it depends only on $T_1$, $T_2$, $\mathbf{v}_1$ and $\mathbf{v}_2$). Same for the other two edges of the triangle and in general for each mesh edge.

**Continuity.** While the basis interpolation $\Psi(\xi_1, \xi_2, \xi_3)$ does not have to be continuous, the corresponding planes do. Formally, continuity at $\Xi$ is:

$$\forall\epsilon > 0 : \exists\delta : \forall\Xi' : \|\Xi' - \Xi\| < \delta \implies d(\Psi(\Xi'), \Psi(\Xi)) < \epsilon.$$

The problem with defining $\Psi(\xi_1, \xi_2, \xi_3) = \xi_1 T_1 + \xi_2 T_2 + \xi_3 T_3$ is that the blend depends on how the bases of the tangent planes are chosen (it may lead to results that belong to different equivalence classes) and can also lead to singularities. The goal is to fix this by choosing the bases intelligently. One cannot choose them globally due to hairy ball theorems, so the bases have to be different for every triangle. The difficulty then is keeping the blending consistent across edges. It is also possible to define a blend using the natural metric on the Grassmannian, but the resulting computations are complicated and expensive; we therefore choose to linearly blend bases.

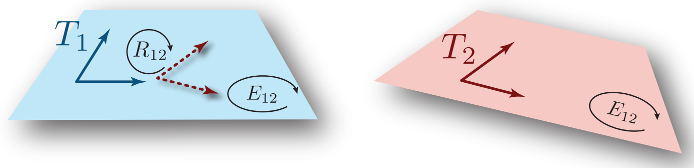



**Figure 1:** The planes at $\mathbf{v}_1$ and $\mathbf{v}_2$ are represented by two orthonormal bases $T_1$ and $T_2$. The orthogonal matrix $R_{12}$ transforms $T_1$ in-plane, relative to $T_2$. Also, both bases $T_1$ and $T_2$ can be simultaneously rotated in-plane by the same orthogonal matrix $E_{12}$. $R_{12}$ and $E_{12}$ are degrees of freedom we can play with to optimize our tangent plane blending $\Psi$. We choose $R_{12}$ such that it transforms $T_1$ to be as close as possible to $T_2$ and we also pick $E_{12}$ to obtain a well-defined blend within the triangle, as described in Section 1.4.

## 1.3 Interpolation on edges

We construct $\Psi$ in two steps: first, we define an interpolant on edges only, and then extend it to the interiors of the triangles.

We start by defining the blends on the edges, i.e., when one of $\xi_1$, $\xi_2$, or $\xi_3$ is zero. To have consistent interpolation between triangles that share an edge, we must carefully pick the bases to blend. Assume, without loss of generality, that $\xi_3 = 0$. We then define $\Psi$ as a blend between $T_1$ and $T_2$ as:

$$\Psi(\xi_1, \xi_2, 0) = \xi_1 R_{12} T_1 + \xi_2 R_{21} T_2, \tag{4}$$

where $R_{12}$ and $R_{21}$ are orthonormal 2-by-2 matrices. Using $R_{12}$ and $R_{21}$ effectively allows us to pick bases for $T_1$ and $T_2$ (see Definition 1.3) that can be linearly blended without introducing singularities. Note that $R_{12}$ and $R_{21}$ are associated with an edge, so $\Psi$ will behave consistently on both triangles that share that edge.

We observe that having both $R_{12}$ and $R_{21}$ is redundant since:

**Observation 1.1.** *Let $T_i$ for $i = 1..n$ be bases (not necessarily orthonormal) for planes and let $A \in \mathbb{R}^{2\times 2}$ be a nonsingular matrix. Let $\xi_i$ be scalar weights. Then we can multiply every $T_i$ by $A$ on the left without changing the plane:*

$$\sum_i \xi_i A T_i = A \sum_i \xi_i T_i \equiv \sum_i \xi_i T_i.$$

**Corollary 1.1.** *If $R_1, R_2 \in \mathbb{R}^{2\times 2}$ are orthogonal matrices and $\xi_1, \xi_2 > 0, \xi_1 + \xi_2 = 1$ then*

$$\xi_1 R_1 T_1 + \xi_2 R_2 T_2 \equiv \xi_1 R_2^T R_1 T_1 + \xi_2 T_2.$$

We can then assume without loss of generality that $R_{21} = I$ and focus on choosing $R_{12}$. We choose it to minimize the difference between the bases (see Figure 1):

$$R_{12} = \mathrm{Ort}(T_2 T_1^T). \tag{5}$$

This choice is motivated by the fact that linearly interpolating bases that are close will always generate a valid basis. Proposition 4.3 shows that this choice is unique for sufficiently close planes, and then the blend on edges is independent of the bases chosen for the tangent planes.

**Definition 1.6.** *For $\xi_1, \xi_2 \geq 0,\ \xi_1 + \xi_2 = 1$, we choose the blend $\Psi(\xi_1, \xi_2, 0)$ such that*

$$\Psi(\xi_1, \xi_2, 0) \equiv \xi_1 \,\mathrm{Ort}(T_2 T_1^T)\, T_1 + \xi_2 T_2.$$

**Proposition 1.1.** *The plane defined by $\Psi(\xi_1, \xi_2, 0)$ does not depend on how $T_1$ and $T_2$ are chosen.*

*Proof.* Recall that we assume that the tangent plane bases at vertices are always chosen to be orthonormal, hence it suffices to check that $\Psi$ always remains in the same equivalence class when $T_1$ and $T_2$ are transformed by some in-plane rotations or reflections. Let $X_1$ and $X_2$ be arbitrary 2-by-2 orthogonal matrices. We have $T_1 \equiv X_1 T_1,\ T_2 \equiv X_2 T_2$. Let us check the blend (Definition 1.6) using these bases:

$$\xi_1 \mathrm{Ort}\left((X_2T_2)(X_1T_1)^T\right)(X_1T_1) + \xi_2(X_2T_2) \overset{\text{Cor. 1.1}}{\equiv} \xi_1 X_2^T \mathrm{Ort}\left(X_2T_2T_1^TX_1^T\right) X_1T_1 + \xi_2 T_2 \overset{\text{Prop. 4.1}}{=}$$
$$= \xi_1 X_2^T X_2 \mathrm{Ort}(T_2T_1^T) X_1^T X_1 T_1 + \xi_2 T_2 = \xi_1 \mathrm{Ort}(T_2T_1^T)T_1 + \xi_2 T_2 \equiv \Psi(\xi_1, \xi_2, 0).$$

□

## 1.4 Interpolation on the triangle interior

So now we know how to blend along edges in a way that only depends on the tangent planes at the edge vertices. We need to extend the blend to the triangle interior in a continuous way. As before, let $T_1$, $T_2$, and $T_3$ be the original orthonormal bases for the tangent planes at the triangle vertices. Let

$$R_{12} = \mathrm{Ort}(T_2T_1^T), \quad R_{23} = \mathrm{Ort}(T_3T_2^T), \quad R_{31} = \mathrm{Ort}(T_1T_3^T).$$

such that

$$\Psi(\xi_1, \xi_2, 0) \equiv \xi_1 R_{12} T_1 + \xi_2 T_2, \quad \Psi(0, \xi_2, \xi_3) \equiv \xi_2 R_{23} T_2 + \xi_3 T_3, \quad \Psi(\xi_1, 0, \xi_3) \equiv \xi_3 R_{31} T_3 + \xi_1 T_1.$$

To define the blend of all three tangent planes in the triangle interior, we extend each edge blend to the interior separately and then blend the three extensions using the weights $1/\xi_1$, $1/\xi_2$ and $1/\xi_3$ (see Figure 2).

To do this, we first need to blend each edge blend with the third tangent plane. For both this blend and the blend between the three extensions, the bases once again need to be consistent, or in other words, the result should not depend on the choice of the bases for $T_1, T_2, T_3$.

Note that in Definition 1.6 we have a degree of freedom per edge in form of a transformation by an orthogonal matrix, i.e., we defined $\Psi(\xi_1, \xi_2, 0)$ up to the equivalence class. Let us denote these degrees of freedom as orthogonal matrices $E_{12}, E_{23}, E_{31} \in \mathbb{R}^{2\times 2}$ for each edge (see Figure 1).

**Definition 1.7.** *Edge blend:*

$$\Psi(\xi_1, \xi_2, 0) = \xi_1 E_{12} R_{12} T_1 + \xi_2 E_{12} T_2,$$

*where the choice of the orthogonal matrix $E_{12} \in \mathbb{R}^{2\times 2}$ will be explained below (in Definition 1.9). The definitions for the other edge blends are analogous.*

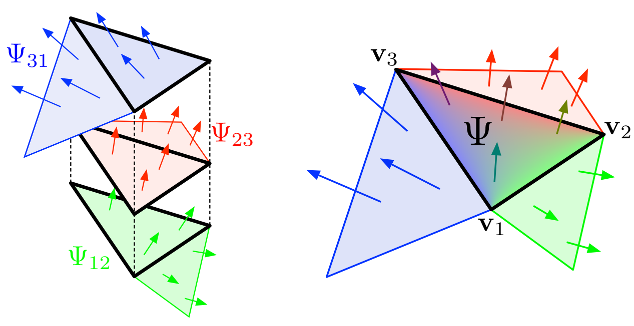



**Figure 2:** We define a continuous blend for each pair of adjacent triangles (flaps) on the mesh (left). To compute $\Psi$ over the black triangle, we blend the interpolants on the overlapping flaps using the weights $1/\xi_1$, $1/\xi_2$ and $1/\xi_3$ (right).

**Definition 1.8.** *Extension of a single edge blend to the triangle interior:*

$$\Psi_{12}(\xi_1, \xi_2, \xi_3) = \xi_1 E_{12} R_{12} T_1 + \xi_2 E_{12} T_2 + \xi_3 \frac{1}{2}(E_{23} + E_{31} R_{31}) T_3.$$

*The definitions for $\Psi_{23}$ and $\Psi_{31}$ are similar.*

We choose the matrices $E_{12}, E_{23}, E_{31}$ so that the matrices: $E_{12}R_{12}T_1$, $E_{12}T_2$, $E_{23}R_{23}T_2$, $E_{23}T_3$, $E_{31}R_{31}T_3$, $E_{31}T_1$ are all as close to each other as possible, such that a linear blend between them would be non-singular.

**Definition 1.9.** *Denote $A_1 = E_{12}R_{12}T_1$, $A_2 = E_{12}T_2$, $A_3 = E_{23}R_{23}T_2$, $A_4 = E_{23}T_3$, $A_5 = E_{31}R_{31}T_3$, $A_6 = E_{31}T_1$. We choose the in-plane transformation per edge as follows:*

$$E_{12}, E_{23}, E_{31} = \underset{E_{12},E_{23},E_{31} \in O(2)}{\operatorname{argmin}} \sum_{1 \le i < j \le 6} \|A_i - A_j\|^2. \tag{6}$$

When two of the $E$'s are fixed, minimizing Equation (6) for the third can be written as a Procrustes problem and solved with the SVD. We can therefore solve for the $E$'s iteratively, one at a time, always decreasing the energy and converging in a few iterations. These calculations need to be performed once per mesh, not per projection.

**Definition 1.10.** *Final blend:*

$$\Psi(\xi_1, \xi_2, \xi_3) = \frac{\xi_1 \xi_2 \xi_3}{\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1} \left( \frac{1}{\xi_3} \Psi_{12}(\xi_1, \xi_2, \xi_3) + \frac{1}{\xi_1} \Psi_{23}(\xi_1, \xi_2, \xi_3) + \frac{1}{\xi_2} \Psi_{31}(\xi_1, \xi_2, \xi_3) \right).$$

The weights in the final blend are chosen as (normalized) $1/\xi_i$, so that we interpolate the edge blends and obtain continuity of $\Psi$ on edges and vertices, as discussed in Section 2. The choice of $E$'s above makes the final result independent of how the original $T_i$'s are chosen and ensures there are no singularities in the blends under reasonable assumptions (shown in Theorem 2.1).

# 2 Continuity of interpolation

While we cannot prove that our plane interpolation is well-defined and continuous unconditionally, we can show that as long as the mesh is a good approximation of a smooth surface, this will be the case.

For a $\mathcal{C}^1$ surface embedded in $\mathbb{R}^D$, the map that takes a point on the surface to its tangent space is continuous in the standard projection operator distance metric and therefore, by Propositions 4.6 and 4.7, in our metric $d$ (see Definition 1.5). Any such surface therefore admits a sufficiently dense triangulation, so that for any triangle, $d(T_i, T_j) < 1/\sqrt{33}$. We expect that much weaker assumptions are possible: we do not attempt to derive the tightest result.

The interpolant over each mesh edge is by construction a linear blend between $T_1$ and $AT_2$ where $A$ is the orthogonal matrix that minimizes $\|T_1 - AT_2\|$. By Proposition 4.3, $A$ is unique.

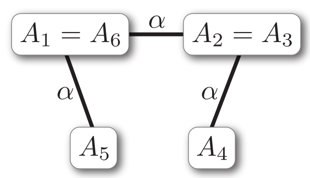
Proposition 1.1 shows that the blended plane does not depend on how $T_1$ and $T_2$ are chosen. Therefore, the interpolated planes over triangles match up on mesh edges and we only need to prove continuity over a single triangle.

**Theorem 2.1.** *If for a triangle, $d(T_i, T_j) < \alpha$, where $\alpha = 1/\sqrt{33}$, then the interpolated plane defined by $\Psi(\Xi)$ is continuous over the set of convex barycentric weights $\Xi$.*

*Proof.* Our blend is a convex combination of six bases: $A_1, \dots, A_6$ (see Definitions 1.9-1.10). The blending weights vary continuously with $\Xi$ except at the triangle vertices (when two out of three $\xi$'s are zero). We therefore need to show two things: **(1)** The planes are continuous at vertices; **(2)** The plane corresponding to the convex combination of the bases is continuous in the blending weights.

**Part 1:** We show that $\Psi(\Xi)$ approaches the plane of $T_1$ as $\Xi \to (1, 0, 0)$ (the other vertices follow by symmetry). The final blend is a convex combination of $\Psi_{12}$, $\Psi_{23}$, and $\Psi_{31}$, and each of these is a convex combination of in-plane rotated bases $T_1$,$T_2$, and $T_3$. As $\xi_2$ and $\xi_3$ approach zero, the coefficients on rotated bases $T_2$ and $T_3$ approach zero in each of the intermediate blends $\Psi_{12}$, $\Psi_{23}$, $\Psi_{31}$ and therefore in the final blend. The coefficient on $\Psi_{23}$, $\xi_2\xi_3/(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1)$ also approaches zero. As $\xi_2$ and $\xi_3$ approach zero, the final blend therefore gets arbitrarily close (in the Frobenius norm) to some convex combination of $E_{12}R_{12}T_1$ and $E_{31}T_1$. Any convex combination of these bases represents the plane $T_1$.

**Part 2:** The blended basis $\Psi(\Xi)$ is a continuous in $\Xi$ (except at the vertices, with which we have just dealt), so the only way the resulting plane can fail to be continuous is if some blended basis is rank-deficient. To show this cannot happen, we need to show that no convex combination of the chosen bases can be rank-deficient. Proposition 4.2 shows that the distance from the matrix $A_1$ (which has orthonormal rows) to the nearest rank-deficient matrix in the Frobenius norm is at least 1. We show that under our assumption, $\forall i\ \|A_1 - A_i\| < 1$ and therefore the distance to the blend in the Frobenius norm is smaller than 1, so the blend cannot be rank-deficient.




We bound the energy achieved in the minimization (6) when solving for the $E$'s. The energy attained by the optimization is going to be no greater than with the following assignment of $E$'s: Setting $E_{12} = I$, $E_{23} = R_{23}^T$, and $E_{31} = R_{12}$, we get $A_2 = A_3$ and $A_1 = A_6$. We also have, $\|A_1 - A_2\| = d(T_1, T_2) < \alpha$ and similarly $\|A_3 - A_4\| < \alpha$ and $\|A_5 - A_6\| < \alpha$. Using the triangle inequality of the Frobenius norm, we have $\|A_4 - A_6\| < 2\alpha$, $\|A_1 - A_4\| < 2\alpha$, $\|A_2 - A_5\| < 2\alpha$, and $\|A_4 - A_5\| < 3\alpha$, etc. Adding up the energy terms, we get

$$\sum_{1\le i<j\le 6} \|A_i - A_j\|^2 < (1^2+1^2+2^2+1^2+0^2+0^2+1^2+2^2+1^2+1^2+2^2+1^2+3^2+2^2+1^2)\alpha^2 = 33\alpha^2 = 1.$$

Therefore, each individual $\|A_1 - A_i\| < 1$. □

# 3 Existence of Phong projection

Although it is possible to construct examples where Phong projection does not exist (see Figure 5 in the paper) we believe it is possible to show that for well-tessellated meshes and for points within (Euclidean) distance $\epsilon$ of the mesh, a Phong projection always exists. The proof strategy could be to focus on the nearest one-ring to the projection point, construct a continuous map from that one-ring to itself whose fixed points are solutions to Equation (1), and use Brouwer's fixed point theorem to deduce the existence of a fixed point. Here is a more detailed informal line of argument:

1. Assume $\mathcal{M}$ is tessellated so that for each vertex $\mathbf{v}$ with tangent plane $T$, the distance $d(T, T') < \alpha$ whenever $T'$ is the tangent plane of a vertex adjacent to $\mathbf{v}$ or the plane of a triangle adjacent to $\mathbf{v}$.

2. Let $\mathbf{p} \in \mathbb{R}^D$ be the point we wish to project onto $\mathcal{M}$ and let $\mathbf{p}'$ be its Euclidean projection onto $\mathcal{M}$; assume $\|\mathbf{p} - \mathbf{p}'\| < \epsilon$.

3. Take the barycentric coordinates of $\mathbf{p}'$ on its mesh face (if it is on an edge or a vertex, it does not matter which face) and let $\mathbf{v}$ be the mesh vertex with the largest barycentric coordinate.

4. Let $O_{\mathbf{v}}$ be $\mathbf{v}$'s one-ring (valence $k$). Let $\Psi$ be a plane in the range of the interpolant over $O_{\mathbf{v}}$. Let $P_\Psi : O_{\mathbf{v}} \to \mathbb{R}^2$ be the projection of $O_{\mathbf{v}}$ onto the plane $\Psi$ through $\mathbf{p}$. Because $\Psi$ is in the range of the interpolant, it is close to the triangle planes and tangent planes of $O_{\mathbf{v}}$. For a fixed $\Psi$, $P_\Psi$ is therefore a homeomorphism of $O_{\mathbf{v}}$ with a polygon in $\mathbb{R}^2$. Moreover, $P_\Psi$ is continuous in $\Psi$ (as $O_{\mathbf{v}}$ varies) and has a continuous inverse over $\Psi$'s image.

5. Consider the map $M : O_{\mathbf{v}} \to O_{\mathbf{v}}$ that takes a point $\mathbf{x}$ on the one-ring and maps it to $P^{-1}_{\Psi(\mathbf{x})}(\mathbf{p})$, where $\Psi(\mathbf{x})$ is the interpolated tangent plane at $\mathbf{x}$. For sufficiently small $\epsilon$, $\mathbf{p}$ is in the image of $P_\Psi$ and $M$ is well-defined. By Theorem 2.1 and by construction of $P_\Psi$, this map is continuous.
6. Apply Brouwer's fixed point theorem to $M$: there exists some $\mathbf{x}$, for which $\mathbf{x} = P^{-1}_{\Psi(\mathbf{x})}(\mathbf{p})$, or, equivalently $\Psi(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{p}) = 0$. Therefore $\mathbf{x}$ satisfies the condition for being a Phong projection.

# 4 Useful results

Below, we use that the Frobenius norm is submultiplicative, namely that for any two matrices $A$ and $B$ that can be multiplied, $\|AB\| \leq \|A\|\|B\|$. The Frobenius norm is rotation-invariant, meaning that if $R^T R = I$, then $\|RA\| = \|A\|$. One of the implications is that if $A$ is an orthonormal 2-by-$D$ basis, then for any matrix or vector $B$ (of height $D$), $\|AB\| \leq \|B\|$.

**Proposition 4.1.** *Let $R, Q$ be (square) orthogonal matrices and $T$ an arbitrary matrix, such that $RT$ and $TQ$ are valid multiplications. Then* $\mathrm{Ort}(RT) = R\,\mathrm{Ort}(T)$ *and* $\mathrm{Ort}(TQ) = \mathrm{Ort}(T)Q$.

*Proof.*

$$\|RT - B\| = \|R(T - R^T B)\| = \|T - R^T B\| \implies \min_{BB^T = I} \|RT - B\| = \min_{BB^T=I} \|T - R^T B\| = \min_{KK^T=I} \|T - K\|,$$

where we substituted $K := R^T B$. Hence the minimum energy attained by $K = \mathrm{Ort}(T)$ and $B = \mathrm{Ort}(RT)$ is the same, and $B = RK$. A similar argument holds for right multiplication $\mathrm{Ort}(TQ) = \mathrm{Ort}(T)Q$. □

We now show that orthogonal matrices and rank-deficient matrices are never close. We use this to argue that if some matrix is close to an orthogonal matrix, it cannot be singular.

**Proposition 4.2.** *Given an $m$-by-$n$ matrix $R$ (with $m \leq n$) such that $RR^T = I$ and a rank-deficient $m$-by-$n$ matrix $A$, the distance between them in the induced 2-norm and therefore the Frobenius norm is at least 1.*

*Proof.*

$$\|R - A\|_2 = \|R - A\|_2 \cdot \|R^T\|_2 \geq \|I - AR^T\|_2 \geq \|\mathbf{x}^T I - \mathbf{x}^T AR^T\| = \|\mathbf{x}\| = 1,$$

where $\mathbf{x}$ is a unit $m$-vector such that $A^T\mathbf{x} = 0$. Recall that $\|M\|_2 \leq \|M\|_F$ for any matrix $M$. □

The next propositions show that aligning bases of different planes is well-defined if they are not too far apart.

**Proposition 4.3.** *Let $T$ and $K$ be orthonormal* 2*-by-*$D$ *bases. If $\|T^T T - K^T K\| < 1$ (i.e., the planes they represent are not too far apart) then $TK^T$ has rank 2.*

*Proof.* Using that multiplication by an orthonormal basis does not increase the Frobenius norm:

$$1 > \|T^T T - K^T K\| \geq \|TT^T TT^T - TK^T KT^T\| = \|I - TK^T (TK^T)^T\|.$$

Since $I$ is orthogonal, by the converse of Proposition 4.2, $TK^T(TK^T)^T$ and therefore $TK^T$ must have full rank. □

It is well-known that $\mathrm{Ort}(A)$ is the orthogonal matrix in the polar decomposition of $A$. If $A$ has full rank, $\mathrm{Ort}(A)$ is unique. Therefore, by Proposition 4.3, for sufficiently close planes, $\mathrm{Ort}(TK^T)$ is well-defined.

**Proposition 4.4.** *If $T$ and $K$ are orthonormal* 2*-by-*$D$ *bases,* $\mathrm{argmin}_{A \in O(2)} \|T - AK\| = \mathrm{Ort}(TK^T)$.

*Proof.* For $A \in O(2)$,

$$\|T - AK\|^2 = \text{tr}\left((T - AK)^T(T - AK)\right) = \text{tr}\left(T^TT - 2K^TA^TT + K^TA^TAK\right) = c - 2\text{tr}\left(K^TA^TT\right),$$

where $c = \text{tr}\left(T^TT + K^TK\right)$ does not depend on $A$. By the same manipulation,

$$\|TK^T - A\|^2 = c' - 2\text{tr}\left(A^TTK^T\right) = c' - 2\text{tr}\left(K^TA^TT\right)$$

(where $c' = \text{tr}\left(KT^TTK^T + I\right)$ again does not depend on $A$) so optimizing $\|T - AK\|$ over $O(2)$ is the same as optimizing $\|TK^T - A\|$ over $O(2)$, the solution to which is given by $\text{Ort}(TK^T)$ by definition. □

**Proposition 4.5.** *The distance $d(T, K)$ is a metric on orthonormal bases modulo $\equiv$.*

*Proof.* Clearly, $d$ is nonnegative and $d(T, K) = 0$ if and only if $T \equiv K$ by definition. Symmetry follows from rotation-invariance of the Frobenius norm: $\|T - AK\| = \|A^TT - K\|$, so if $A$ attains the minimum for $d(T, K)$ then $A^T$ attains the minimum for $d(K, T)$. It remains to show the triangle inequality. For some $A$ and $A'$,

$$d(T, K) + d(K, M) = \|T - AK\| + \|K - A'M\|.$$

Then

$$d(T, K) + d(K, M) = \|T - AK\| + \|AK - AA'M\| \geq \|T - AA'M\| \geq d(T, M).$$

□

Now we can show in the following two propositions that the projection operator distance $\|T^TT - K^TK\|$ and our $d(T, K)$ are equivalent up to a constant.

**Proposition 4.6.** *If $T$ and $K$ are orthonormal 2-by-$D$ bases,*

$$d(T, K) \geq \frac{1}{2}\|T^TT - K^TK\|.$$

*Proof.* We need to show that for any orthogonal 2-by-2 matrix $A$, $2\|T - AK\| \geq \|T^TT - K^TK\|$. Because the Frobenius norm is rotation-invariant, we have $\|T - AK\| = \|A^TT - A^TAK\| = \|T^TA - K^T\|$. Multiplication by a 2-by-$D$ orthonormal basis does not increase the norm, so:

$$\|T - AK\| \geq \|T^TT - T^TAK\|, \qquad \|T^TA - K^T\| \geq \|T^TAK - K^TK\|.$$

Adding these inequalities and using the triangle inequality, we obtain:

$$2\|T - AK\| \geq \|T^TT - T^TAK\| + \|T^TAK - K^TK\| \geq \|T^TT - K^TK\|.$$

□

**Proposition 4.7.** *If $T$ and $K$ are orthonormal 2-by-$D$ bases,*

$$d(T, K) \leq \sqrt{2}\|T^TT - K^TK\|.$$

*Proof.* By proposition 4.4, $d(T, K) = \|T - \text{Ort}(TK^T)K\|$. Because $T$ is orthonormal, multiplication by it cannot increase the norm, so $\|T^TT - K^TK\| \geq \|T - TK^TK\|$. Therefore, it suffices to show that

$$\|T - \text{Ort}(TK^T)K\|^2 \leq 2\|T - TK^TK\|^2. \tag{7}$$

We write the left hand side in terms of traces, using that $\text{tr}(TT^T) = 2$ and letting $U\Sigma V^T$ be the SVD of $TK^T$:

$$\begin{aligned}\|T - \text{Ort}(TK^T)K\|^2 &= \text{tr}\left((T - \text{Ort}(TK^T)K)(T - \text{Ort}(TK^T)K)^T\right) = \\ &= \text{tr}\left(TT^T - 2\text{Ort}(TK^T)KT^T + \text{Ort}(TK^T)KK^T\text{Ort}(TK^T)^T\right) = \\ &= 4 - 2\text{tr}(\text{Ort}(TK^T)KT^T) = 4 - 2\text{tr}(UV^TV\Sigma U^T) = 4 - 2\|\Sigma\|_1\end{aligned}$$

Similarly, with the right hand side of (7):

$$2\|T - TK^TK\|^2 = 2\mathrm{tr}\left((T - TK^TK)(T - TK^TK)^T\right) = 2\mathrm{tr}\left(TT^T - 2TK^TKT^T + TK^TKK^TKT^T\right) =$$
$$= 4 + 2\mathrm{tr}\left(-2TK^TKT^T + TK^TKT^T\right) = 4 - 2\mathrm{tr}\left(TK^TKT^T\right) = 4 - 2\|\Sigma\|^2.$$

The singular values of $T$ and $K^T$ are no greater than one and therefore, the singular values of $TK^T$ are no greater than one. This implies that $4 - 2\|\Sigma\|_1 \leq 4 - 2\|\Sigma\|^2$ as needed. $\square$

# 5 Moving least squares and our weights

Given $n$ points $\mathbf{y}_1, \ldots, \mathbf{y}_n$ in $\mathbb{R}^D$ and an $n$-by-$n$ diagonal matrix of weights $\mathbf{D}$, our generalized barycentric coordinates $\mathbf{w}$ at the origin (without the projection constraint) are defined as the argmin of $\|\mathbf{Dw}\|^2$ subject to $\mathbf{Yw} = \mathbf{0}$ and $\mathbf{1}^T\mathbf{w} = 1$, where $\mathbf{1}$ is an $n$-by-1 matrix of ones, $\mathbf{Y}$ is a $D$-by-$n$ concatenation of $\mathbf{y}_i$'s. Letting $\mathbf{q} = \mathbf{Dw}$, the objective becomes $\|\mathbf{q}\|^2$ and the constraint becomes

$$\begin{pmatrix} \mathbf{Y} \\ \mathbf{1}^T \end{pmatrix} \mathbf{D}^{-1}\mathbf{q} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}. \tag{8}$$

Solving the constrained minimization is equivalent to finding the minimum-norm solution of the linear system defined by the constraints: $\mathbf{w} = \mathbf{D}^{-1}((\mathbf{Y}^T \;\; \mathbf{1})^T\mathbf{D}^{-1})^+(\mathbf{0}^T \;\; 1)^T$, where $(\cdot)^+$ is the pseudoinverse. This is just the last column of $\mathbf{D}^{-1}((\mathbf{Y}^T \;\; \mathbf{1})^T\mathbf{D}^{-1})^+$.

Given values $z_i$ associated with points $\mathbf{y}_i$, the moving least squares (with a linear basis) interpolant at $\mathbf{y} = \mathbf{0}$ is the value of the planar function $a + \mathbf{y}^T\mathbf{b}$ that minimizes the energy:

$$\sum_{i=1}^{n} \phi(\mathbf{y}_i)^2 \left(a + \mathbf{y}^T\mathbf{b}_i - z_i\right)^2$$

where $\phi(\mathbf{y})$ is a kernel function, like $\phi(\mathbf{y}) = 1/\|\mathbf{y}\|^2$. Written in matrix form, the energy is:

$$\|\mathbf{D}^{-1}\left(\mathbf{Y}^T\mathbf{b} + \mathbf{1} \cdot a\right) - \mathbf{D}^{-1}\mathbf{z}\|^2$$

where $\mathbf{z}$ is an $n$-by-1 stack of $z_i$'s, and $\mathbf{D}$ is a diagonal matrix of $1/\phi(y_i)$'s. The solution is

$$\begin{pmatrix} \mathbf{b} \\ a \end{pmatrix} = (\mathbf{D}^{-1}(\mathbf{Y}^T \;\; \mathbf{1}))^+\mathbf{D}^{-1}\mathbf{z}.$$

At $\mathbf{0}$, the value of $a + \mathbf{y}^T\mathbf{b}$ is just $a$, so the weights on $\mathbf{z}$ are the last row of $(\mathbf{D}^{-1}(\mathbf{Y}^T \;\; \mathbf{1}))^+\mathbf{D}^{-1}$. This is exactly the transpose of our weights.

# References


[1] Daniele Panozzo, Ilya Baran, Olga Diamanti, and Olga Sorkine-Hornung. Weighted averages on surfaces. *ACM Trans. Graph. (Proc. ACM SIGGRAPH 2013)*, 31(4), 2013.