

# Why Linear Algebra?

(for Computer Scientists)

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Olga Sorkine-Hornung

# Why Math?

- Computer Scientists,  
not just programmers
- Practice solid arguments,  
correctness proofs -  
it's an art!



# Main topics of the LA class

- Linear systems of equations

$$3x_1 + 4x_2 - 1.5x_3 = 0$$

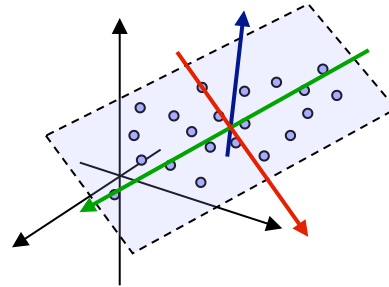
$$x_1 - 3.2x_2 + 5x_3 = 17$$

$$2x_1 + 7x_2 + 3.1x_3 = 42$$

$$A\mathbf{x} = \mathbf{b}$$

- Linear (vector) spaces and transformations

$$\mathbf{x}, \mathbf{y} \in V \Rightarrow \alpha\mathbf{x} + \beta\mathbf{y} \in V$$



# Linear Algebra is everywhere

- Most world's phenomena involve complicated equations
- Computers can only do basic arithmetic
- → Usually can't do the original equations, approximate by series of **linear equations**
- → Model things as **linear spaces**

$$E(q) = \sum_{nT} A_T \|\nabla q^T - \mathbf{w}^T\|^2 \rightarrow \min$$

$$E(V') = \sum_{i=1} \|\delta_i - \mathcal{L}(\mathbf{v}'_i)\|^2 + \sum_{i=m}^n \|\mathbf{v}'_i - \mathbf{u}_i\|^2,$$

$$q_m^i = \sum_k \frac{1}{M_i - 1} \sum_{j \in O_k} X^{ji} \sum_{n=1}^3 w_{mn}^{ij} q_n^j$$

$$\arg \min_{w_j} \frac{1}{2} \int_{\Omega} |\Delta w_j|^2 dV$$

$$E(q) = \sum_T A_T \|\nabla q^T - \mathbf{w}^T\|^2 \rightarrow \min$$

$$\tilde{I}^i(\cdot) : \bigcup_{k=1}^{d_i-1} \Delta_k^i \rightarrow \mathbb{R}.$$

$$\tilde{I}^i(\mu) = \langle \mu, \mu \rangle_{\mathbb{R}^3} = \langle \mu_1 \tilde{\mathbf{x}}_k^i + \mu_2 \tilde{\mathbf{x}}_{k+1}^i, \mu_1 \tilde{\mathbf{x}}_k^i + \mu_2 \tilde{\mathbf{x}}_{k+1}^i \rangle_{\mathbb{R}^3} =$$

$$= \mu_1^2 \tilde{g}_{k,k}^i + 2 \mu_1 \mu_2 \tilde{g}_{k,k+1}^i + \mu_2^2 \tilde{g}_{k+1,k+1}^i,$$

$$\hat{\mathbf{x}}_2^i = \langle \tilde{\mathbf{x}}_2^i, \frac{\tilde{\mathbf{x}}_1^i}{\|\tilde{\mathbf{x}}_1^i\|} \rangle \frac{\tilde{\mathbf{x}}_1^i}{\|\tilde{\mathbf{x}}_1^i\|} + \langle \tilde{\mathbf{x}}_2^i, \mathbf{n} \rangle \mathbf{n} + \langle \tilde{\mathbf{x}}_2^i, \mathbf{N}^i \rangle \mathbf{N}^i =$$



# Examples from everyday life

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# Weather forecasting

- Solve PDEs (partial differential equations) that model the physics of the atmosphere
- Unknowns: temperature, humidity, wind... at every point in Earth's atmosphere at a certain time

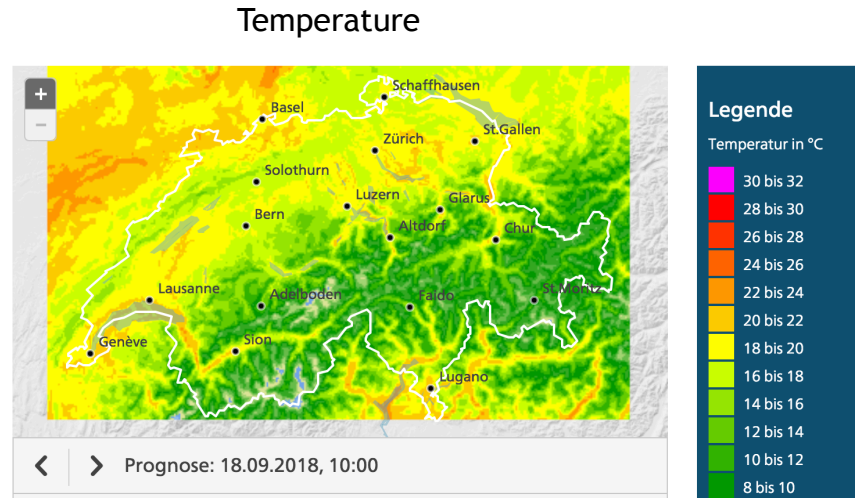


Image source: MeteoSwiss

# Weather forecasting

- Analytical solution (formula) doesn't exist
- → Discretization on a grid, numerical approximation
- Huge systems of linear equations

$$Ax = b$$

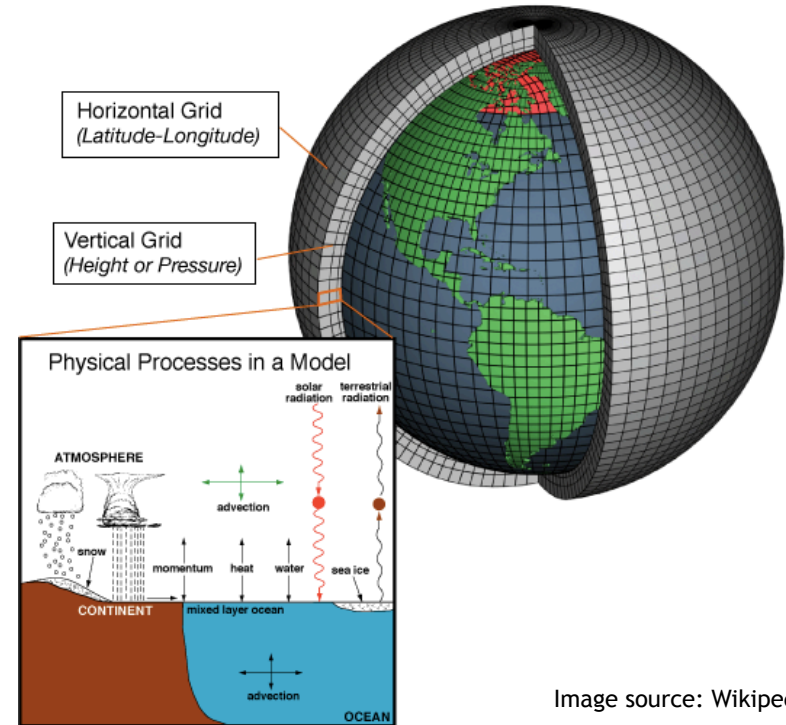


Image source: Wikipedia

# Weather forecasting

- Linear algebra done by supercomputers!
- CS challenge: how to solve huge linear equations, and fast



Some of the MeteoSwiss supercomputers at CSCS, Lugano

# Google search engine

- Web crawler “reads” the Internet pages and indexes by keywords
- User enters keyword, search engine retrieves pages containing it
- In what order to present the found pages??



Lineare Algebra ETH



Google Search

I'm Feeling Lucky

[Lineare Algebra I & II, Studienjahr 2017/2018 - ETH Zürich](https://metaphor.ethz.ch/x/2017/hs/401-1151-00L/)

<https://metaphor.ethz.ch/x/2017/hs/401-1151-00L/> [Translate this page](#)

4.5: Endomorphismen und Determinanten, Woche 13, [FIS] § 4.5; [F] §3.4; [J] § 6.7; [P] § 6.5. Ende des Prüfungstoffes Lineare Algebra I. § 5.1: Eigenwerte und ...

[igl | Interactive Geometry Lab | ETH Zurich | Linear Algebra HS 2017 ...](http://igl.ethz.ch/teaching/linear-algebra/la2017/)

[igl.ethz.ch/teaching/linear-algebra/la2017/](http://igl.ethz.ch/teaching/linear-algebra/la2017/) [Translate this page](#)

May 9, 2018 - **Lineare Algebra**. Orthogonal projection; best rigid fit. Vorlesungs-Nr. 401-0131-00; Semester: Herbst 2017; Dozenten: Özlem Imamoglu,

[Lineare Algebra Herbst 2017 - ETH Zürich](https://metaphor.ethz.ch/x/2017/hs/401-0151-00L/)

<https://metaphor.ethz.ch/x/2017/hs/401-0151-00L/> [Translate this page](#)

September (2. Semesterwoche), wird im Anschluss an die Vorlesung das Buch zur Vorlesung verkauft: K. Nipp/D. Stoffer, **Lineare Algebra**, vdf Hochschulverlag, ...

[ETH :: D-MATH :: Lineare Algebra I](http://www2.math.ethz.ch/education/bachelor/lectures/.../linalg1.html)

[www2.math.ethz.ch/education/bachelor/lectures/.../linalg1.html](http://www2.math.ethz.ch/education/bachelor/lectures/.../linalg1.html) [Translate this page](#)

Präsenz: Ab der vierten Semesterwoche mittwochs, 12:00 - 13:00 im HG J 15.1. Zwischenprüfung. Die Zwischenprüfungsnoten wurden versandt. Wenn Sie Ihre ...

# Google search engine - PageRank

- PageRank algorithm sorts search results by importance
- Importance of a page = how many other important pages link to it

$$\text{PageRank}(u) = \sum_{v: v \text{ links to } u} \frac{\text{PageRank}(v)}{\# \text{ links from } v}$$

- PageRanks of all webpages?  
Eigenvalue problem!  $A\mathbf{u} = \lambda\mathbf{u}$
- We will learn about it in the 2<sup>nd</sup> half of the semester

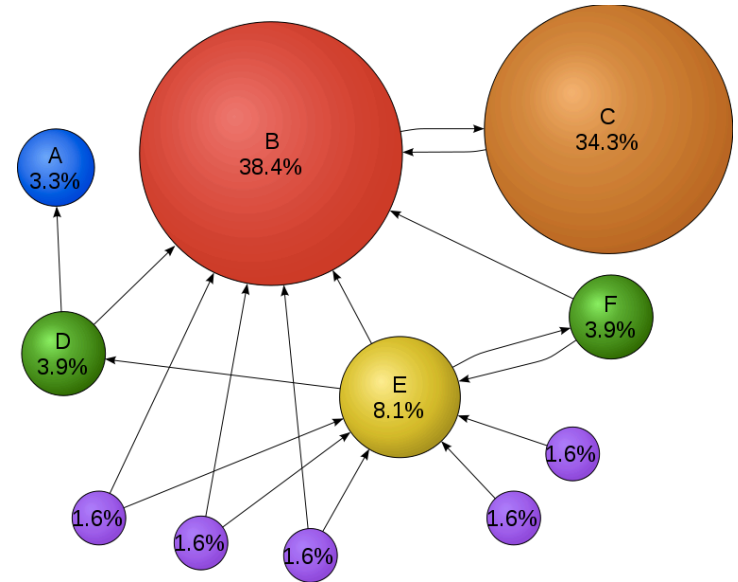


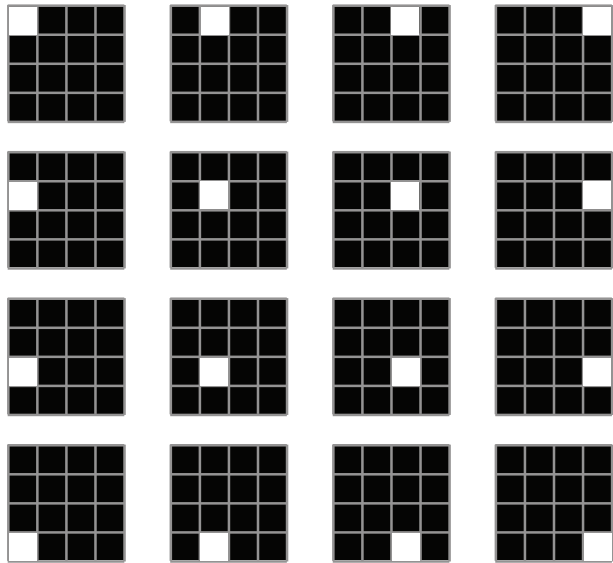
Image source: Wikipedia

# Digital image representation

- Images are vectors!
- The image on the right:
  - 2272 x 1704 pixels
  - pixel = (R,G,B)-value
  - this image is a **11,614,464-dimensional vector**



# Images as vectors



The standard basis for 4x4 grayscale images  
16 vectors

Any 4x4 grayscale image is a  
linear combination of this standard basis

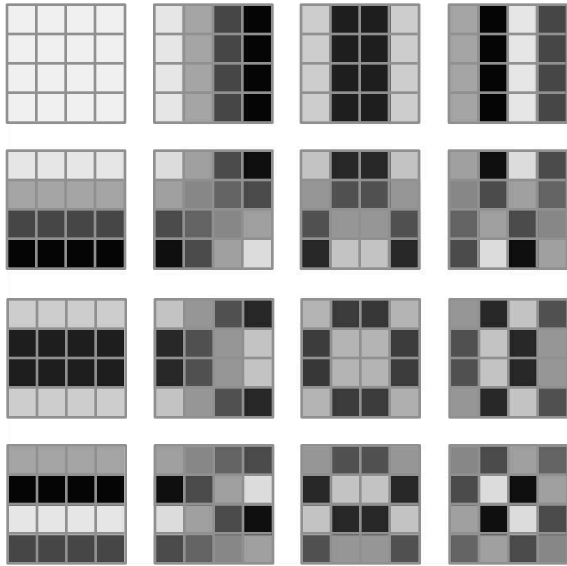
$$= 1 * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 0.6 * \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \dots$$

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \dots + \alpha_n \mathbf{b}_n$$

Need to store all  $\alpha_1, \alpha_2, \dots, \alpha_n$



# JPEG image compression



The 4x4 DCT (discrete cosine) basis  
16 vectors

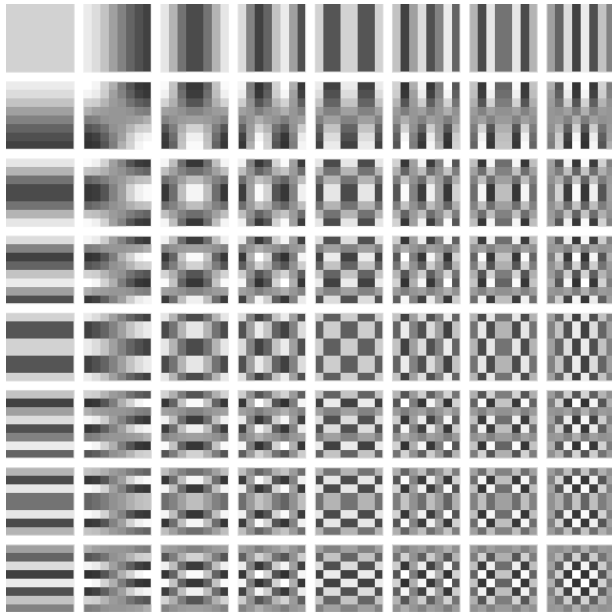
Any 4x4 grayscale image is also a  
linear combination of that basis!

$$\begin{array}{|c|c|c|c|} \hline \text{Image} & = & 1.1 * & \begin{array}{|c|c|c|c|} \hline \text{Basis 1} & \hline \hline \hline \hline \end{array} & + & 0.9 * & \begin{array}{|c|c|c|c|} \hline \text{Basis 2} & \hline \hline \hline \hline \end{array} & + & \dots \\ \hline \end{array}$$

$$\mathbf{x} = \beta_1 \mathbf{c}_1 + \beta_2 \mathbf{c}_2 + \dots + \beta_n \mathbf{c}_n$$

For “natural” images we can omit  
all but a few first  $\beta$

# JPEG image compression



The 8x8 DCT (discrete cosine) basis  
64 vectors

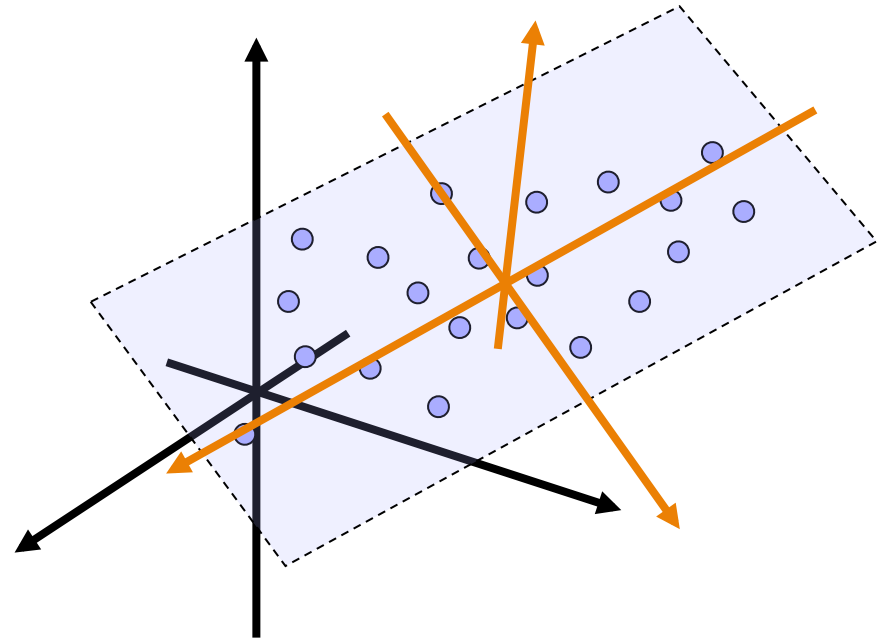
Any 8x8 grayscale image is a  
linear combination of that basis!

$$\mathbf{x} = \beta_1 \mathbf{c}_1 + \beta_2 \mathbf{c}_2 + \dots + \beta_n \mathbf{c}_n$$

For “natural” images we can omit  
all but a few first  $\beta$

# JPEG image compression

- Images are vectors in a (high-dimensional) space
- Different coordinate systems = different bases
- JPEG image compression: project onto a lower-dimensional linear space



# Computer animation

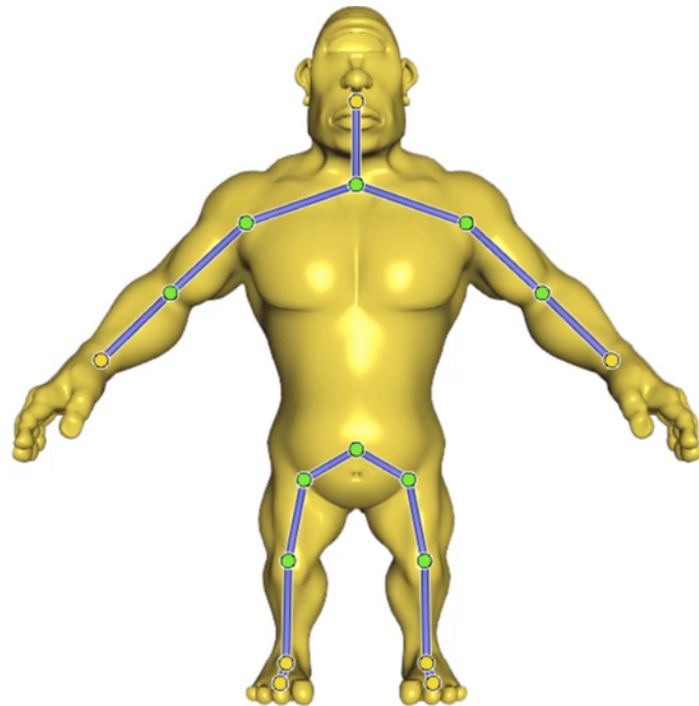
- How do virtual characters move?



Excerpt from “Big Buck Bunny”, open Blender movie

# Computer animation

- Artist designs key poses for skeleton
- Collection of linear transformations in 3D space
- Automatic interpolation over the character's surface and over time



# Linear Algebra is fundamental

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## Enjoy the class!

# Interactive sessions: Clicker

- Install the **ETH EduApp** on your smartphone
  - or -
- Make sure you can log in at <https://eduapp-app1.ethz.ch/>

