252-0538-00L, Spring 2025

Shape Modeling and Geometry Processing

Introduction and Overview

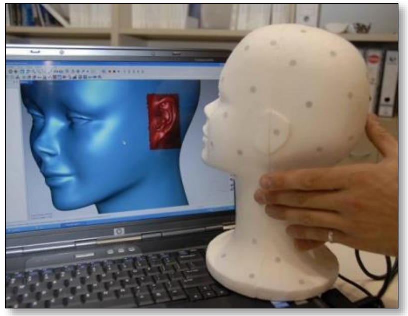
INTERACTIVE GEOMETRY LAB

February 19, 2025

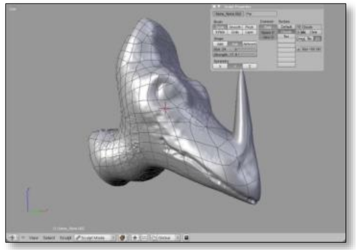
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Shape Representation: Origin- and Application-Dependent

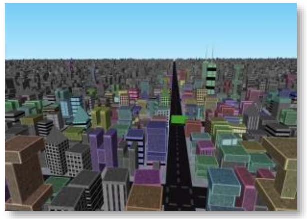
- Acquired real-world objects:
  - Discrete sampling
  - Points, meshes

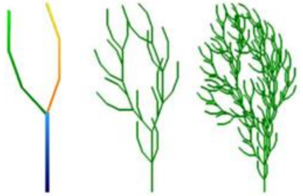



- Modeling “by hand”:
  - Higher-level representations, amendable to modification, control
  - Parametric surfaces, subdivision surfaces, implicits

- Procedural modeling
  - Algorithms, grammars

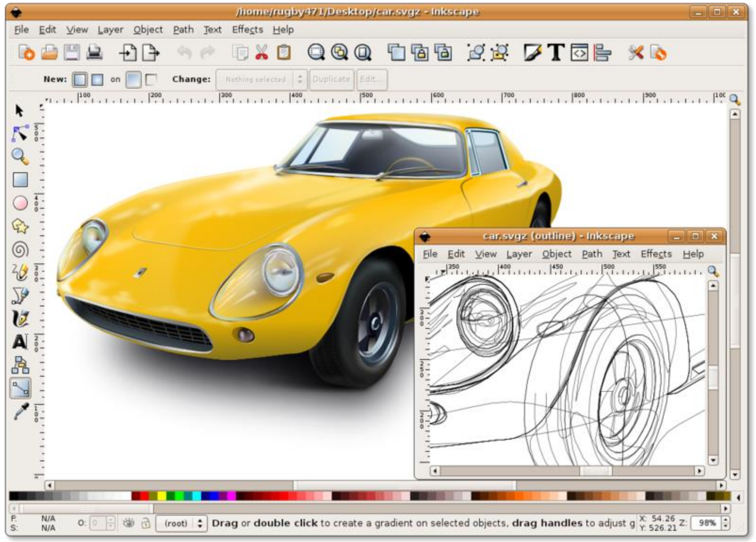


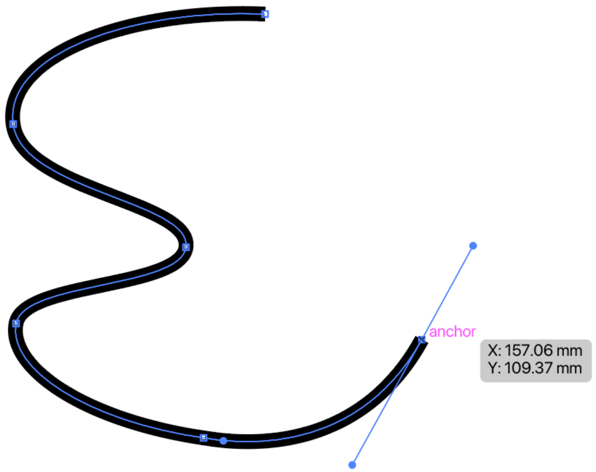


Similar to the 2D Image Domain

- Acquired digital images:
  - Discrete sampling
  - Pixels on a grid

- Painting “by hand”:
  - Strokes + color/shading
  - Vector graphics
  - Controls for editing





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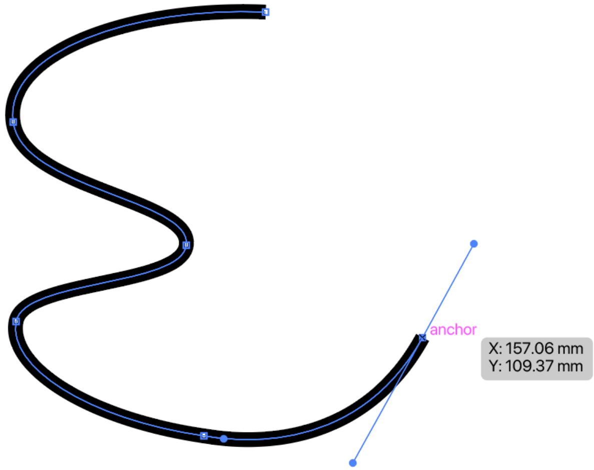
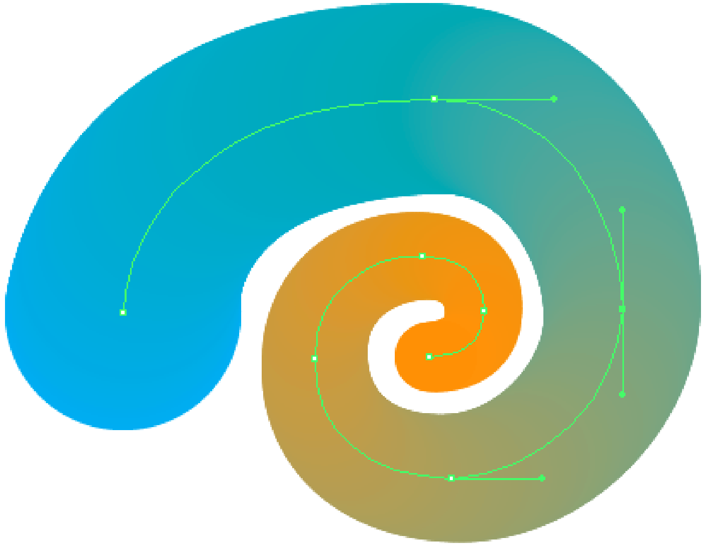
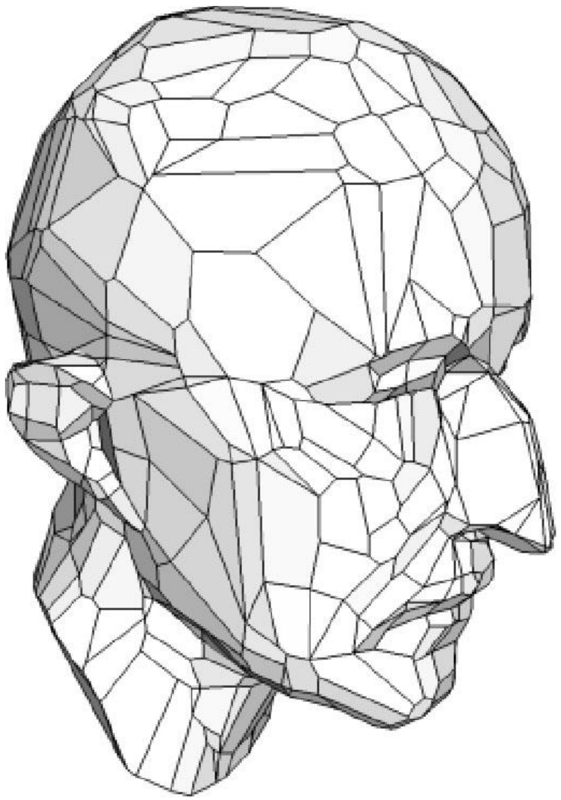




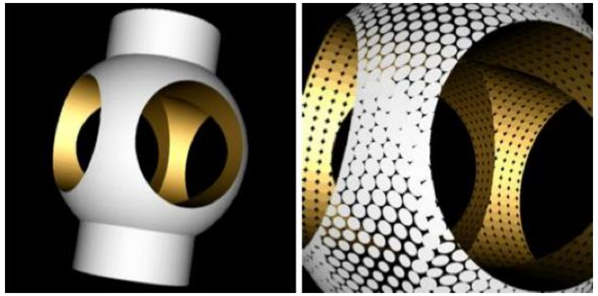
Representation Considerations

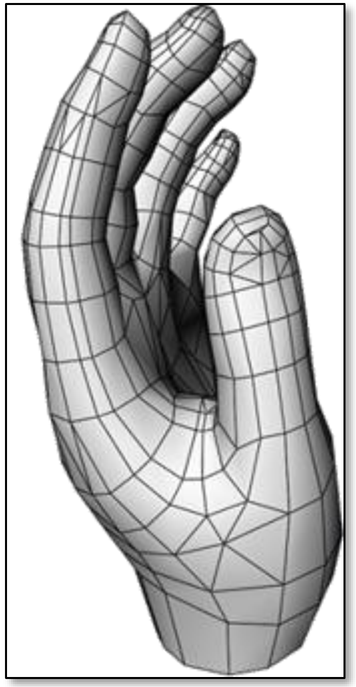
- How should we represent geometry?
  - Needs to be stored in the computer
  - Creation of new shapes
    - Input metaphors, interfaces, generative AI …
  - What operations do we apply?
    - Editing, simplification, smoothing, filtering, repair…
  - How to render it?
    - Rasterization, raytracing…

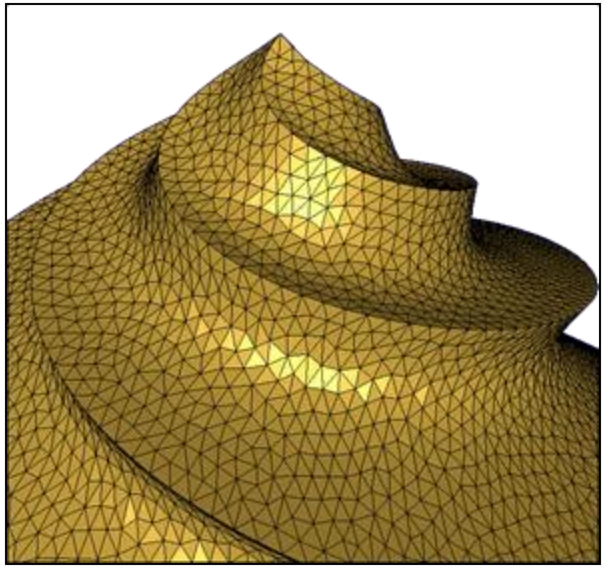


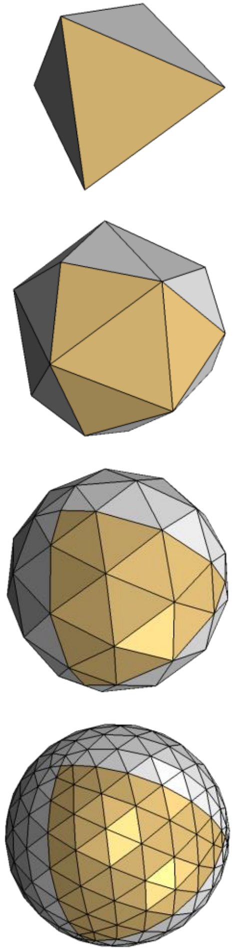
Shape Representations

- Points
- Polygonal meshes



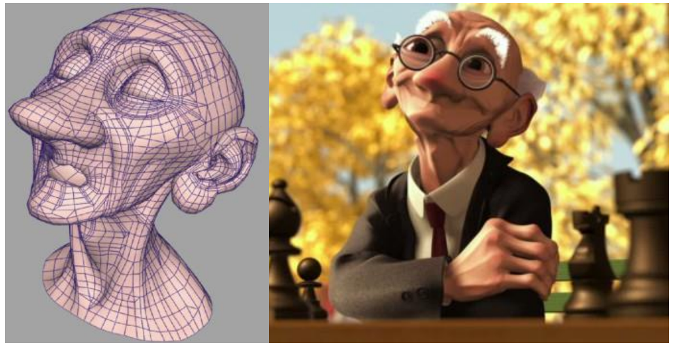


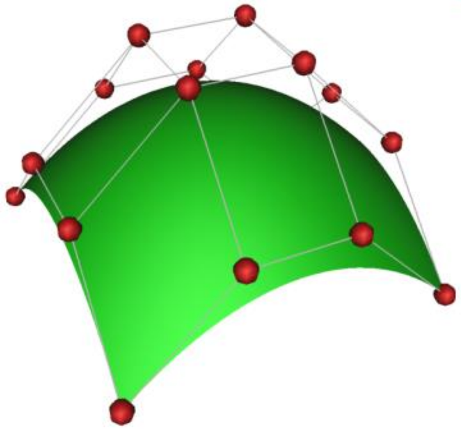


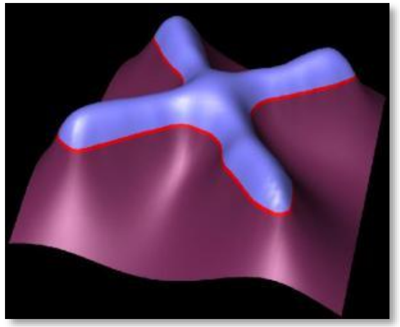


Shape Representations

- Parametric surfaces
- Implicit functions
- Subdivision surfaces







Points

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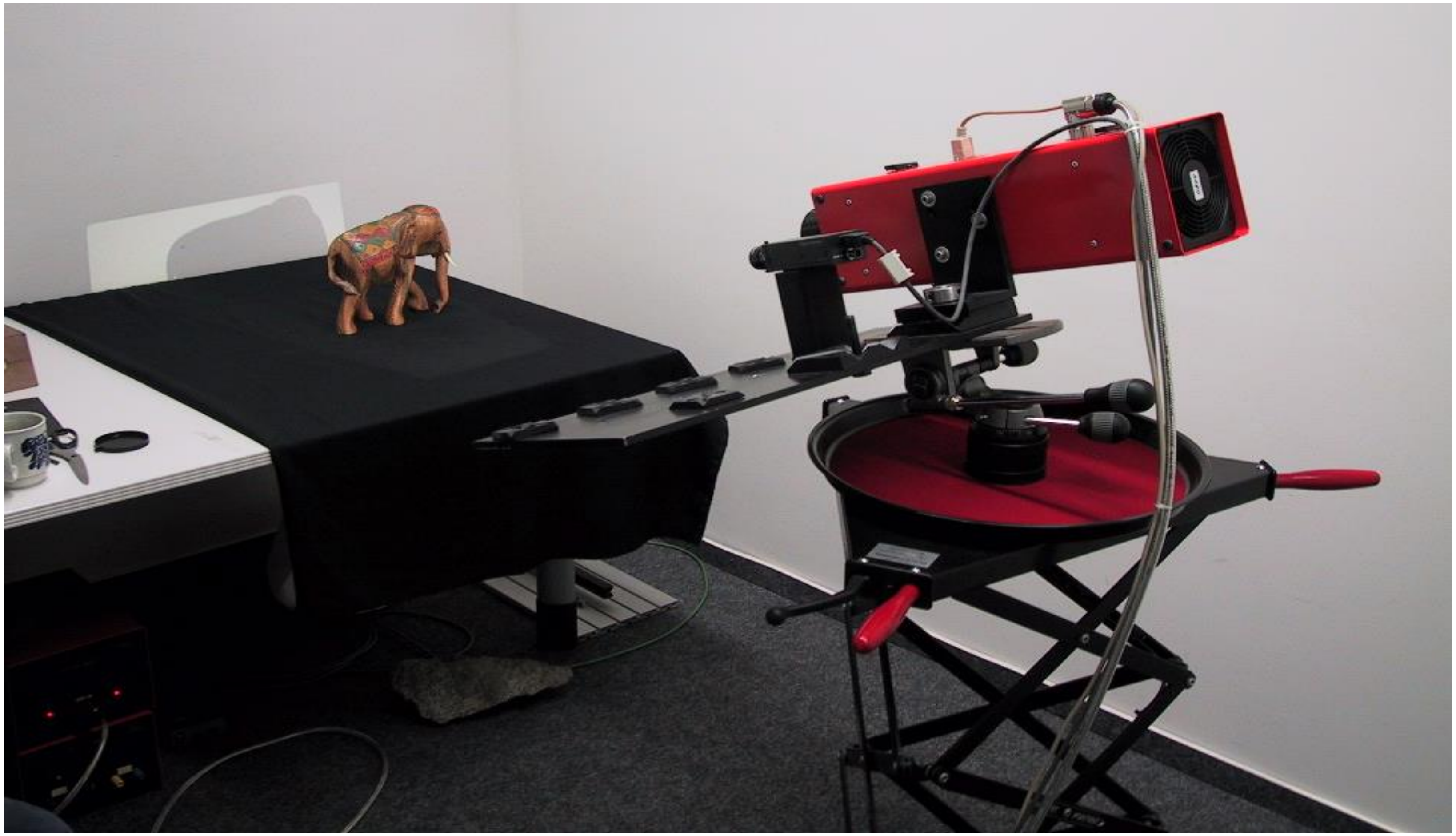
Output of Acquisition



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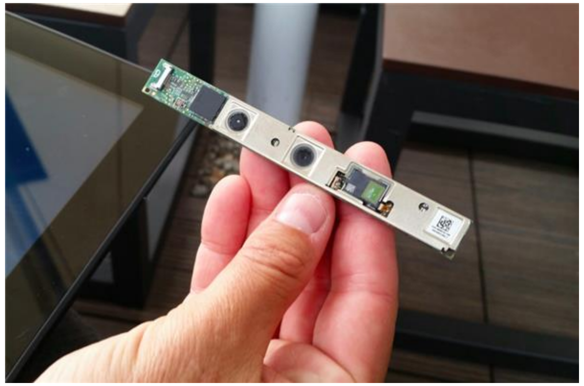
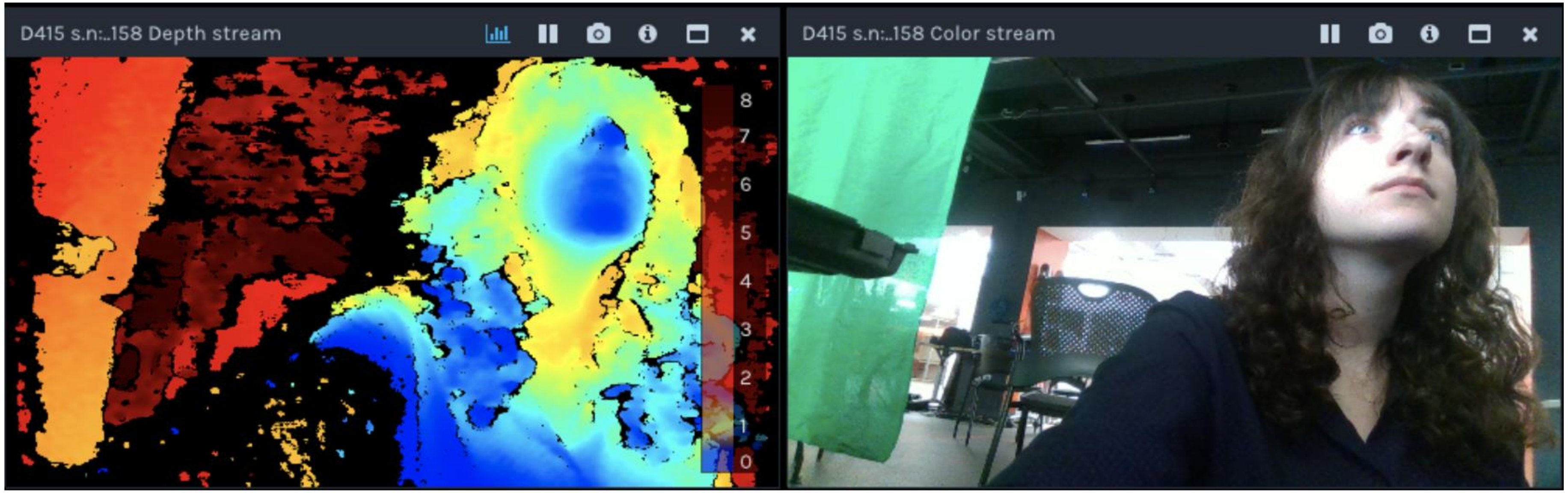
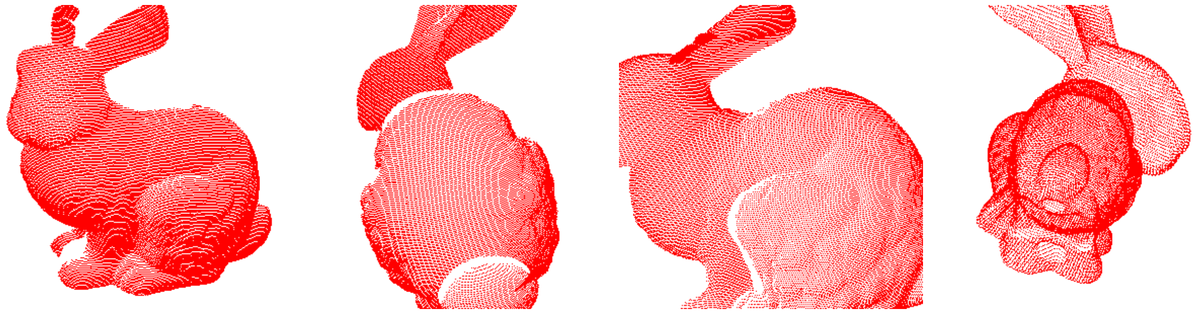


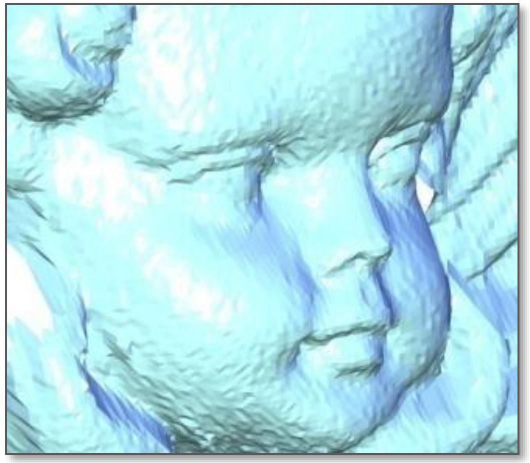


https://frl.nyu.edu/volumetric-capture-with-realsense-depth-cameras/

Points

- Standard 3D data from a variety of sources
  - Often results from scanners
  - Potentially noisy

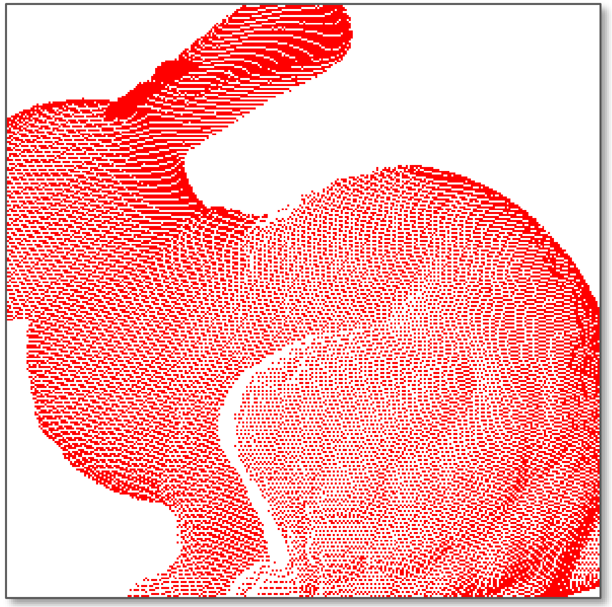




  - Depth imaging (e.g. by triangulation)
  - Registration of multiple images

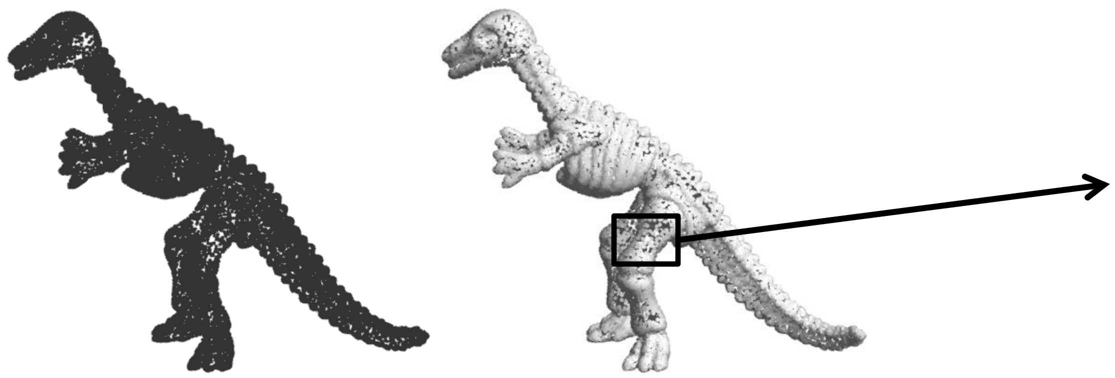
Points

- points = unordered set of 3-tuples
- Often converted to other reps
  - Meshes, implicits, parametric surfaces
  - Those are easier to process, edit and/or render
- Spatial partitioning data structure required
  - For efficient point processing
  - To figure out neighborhoods

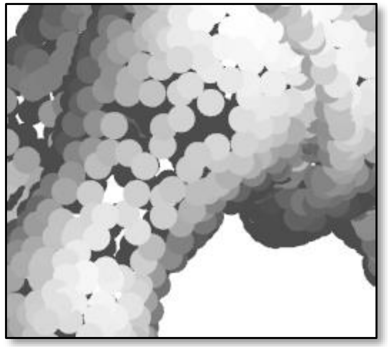


Points: Neighborhood information

- Why do we need neighbors?



need normals (for shading)

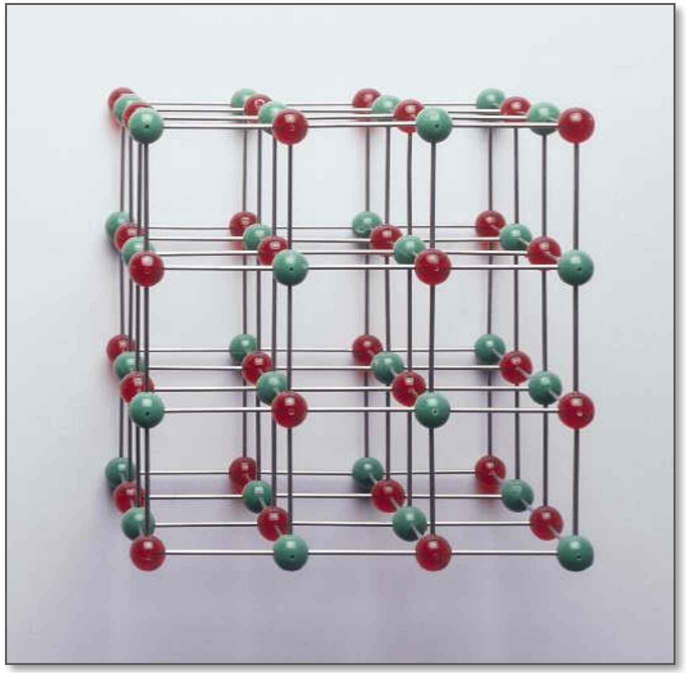


upsampling – need to count density

- Need sub-linear-time implementations of
  - k-nearest neighbors to point $\mathbf{x}$
  - in-radius search $\|\mathbf{p}_i - \mathbf{x}\| < \varepsilon$

Spatial Data Structures

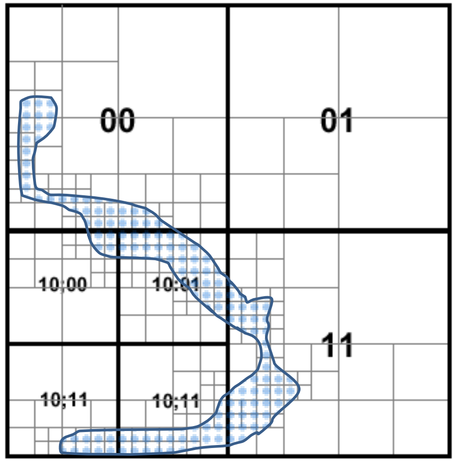
- Regular uniform 3D lattice
  - Simple point insertion by coordinate discretization
  - Simple proximity queries by searching neighboring cells

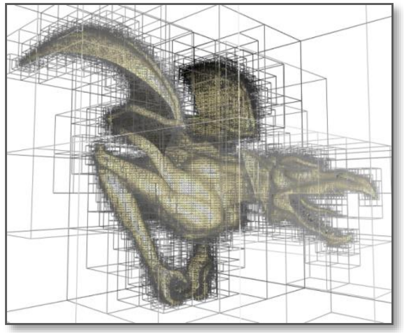


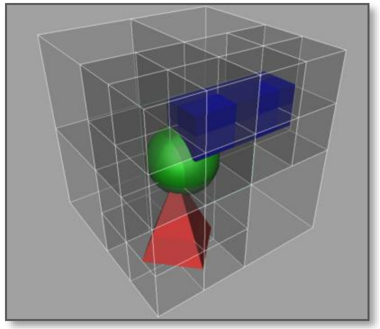
  - Determining lattice parameters (i.e. cell dimensions) is nontrivial
  - Generally unbalanced, i.e. many empty cells

Spatial Data Structures

- Octree
  - Splits each cell into 8 equal cells
  - Adaptive, i.e. only splits when too many points in cell
  - Proximity search by (recursive) tree traversal and distance to neighboring cells
  - Tree might be unbalanced

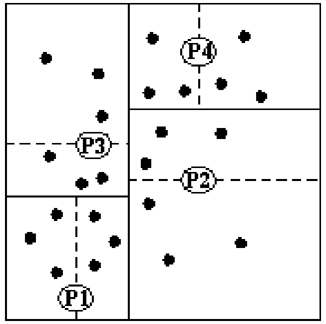


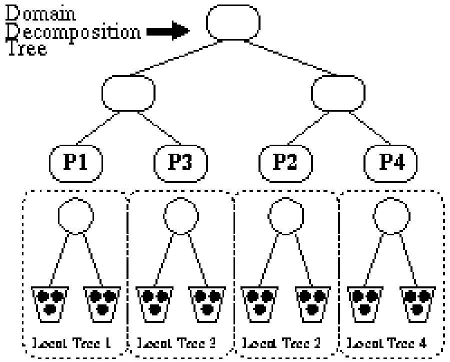


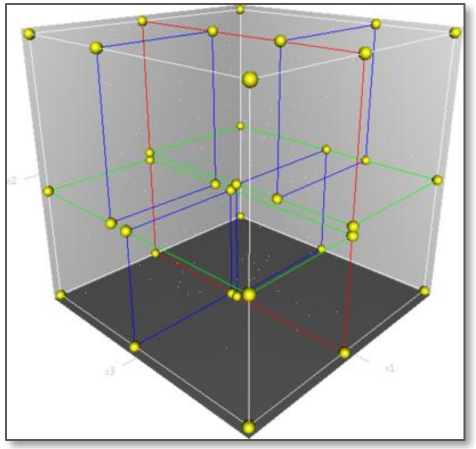


Spatial Data Structures

- Kd-Tree
  - Each cell is individually split along the **median** into two cells
  - Same amount of points in cells
  - Perfectly balanced tree
  - Proximity search similar to the recursive search in an Octree
  - More data storage required for inhomogeneous cell dimensions







Parametric Curves and Surfaces

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Parametric Representation

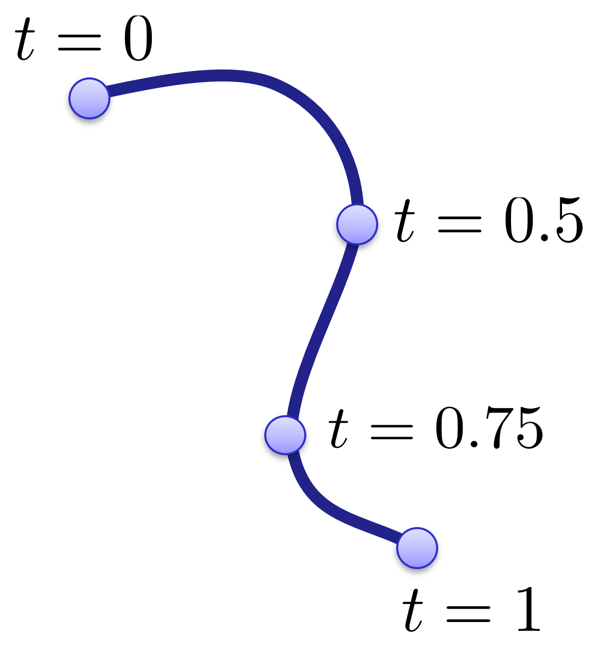
- **Range** of a function $f : X \to Y, \quad X \subseteq \mathbb{R}^m, \; Y \subseteq \mathbb{R}^n$

  - Planar curve: $m = 1, n = 2$

    $$f(t) = (x(t), y(t))$$

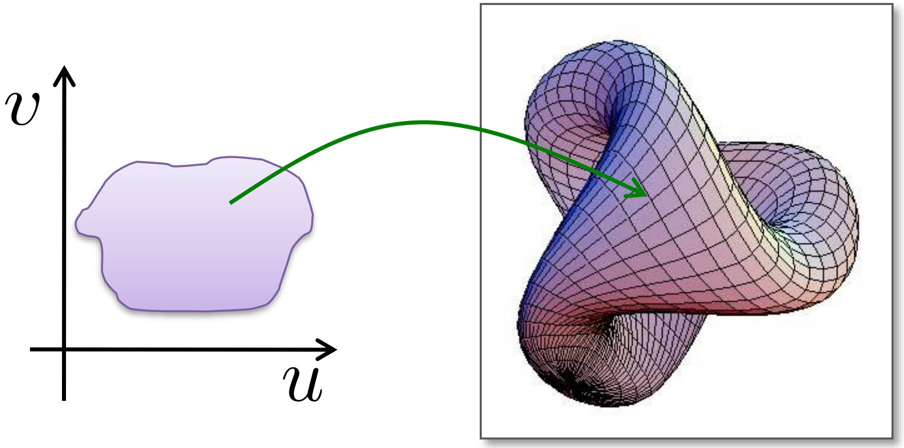
  - Space curve: $m = 1, n = 3$

    $$f(t) = (x(t), y(t), z(t))$$



Parametric Representation

- **Range** of a function $f : X \to Y, \quad X \subseteq \mathbb{R}^m, \ Y \subseteq \mathbb{R}^n$
  - Surface in 3D: $m = 2, n = 3$



$$f(u,v) = (x(u,v), y(u,v), z(u,v))$$

Parametric Curves

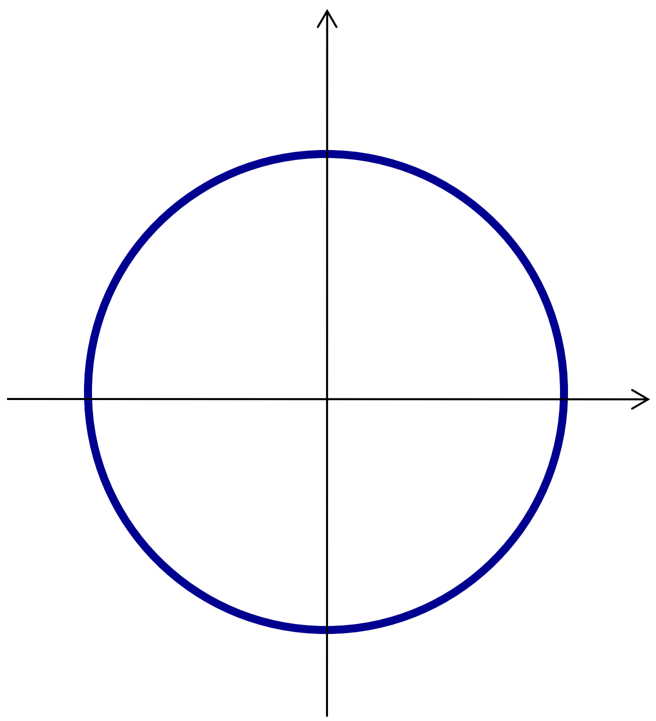
- Parametric circle in 2D

$$\mathbf{p}: \mathbb{R} \to \mathbb{R}^2$$

$$t \mapsto \mathbf{p}(t) = (x(t), y(t))$$

$$\mathbf{p}(t) = r\,(\cos(t), \sin(t))$$

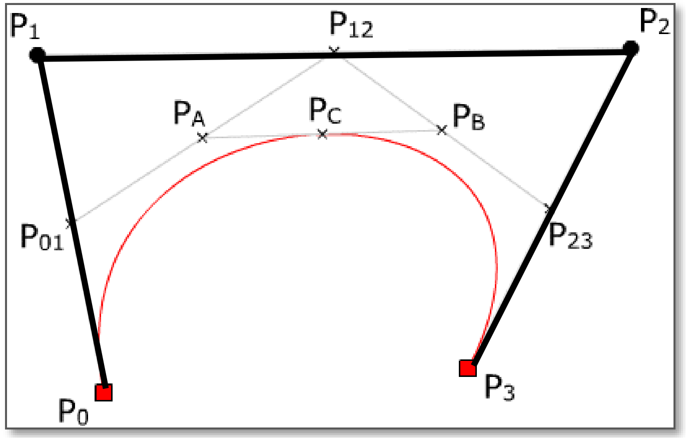
$$t \in [0, 2\pi)$$



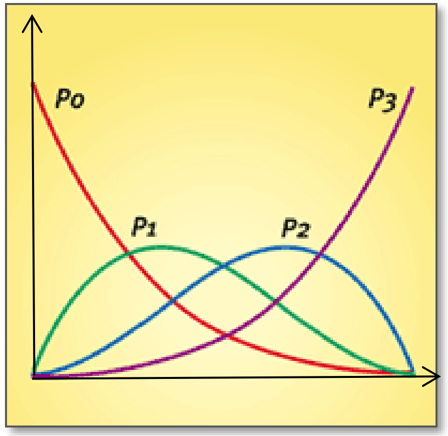
Parametric Curves

- Bézier curves, splines

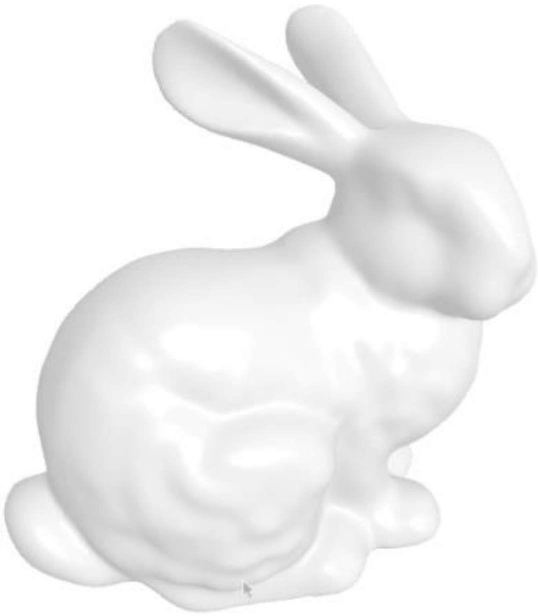
$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^n(t) \qquad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



Curve and control polygon



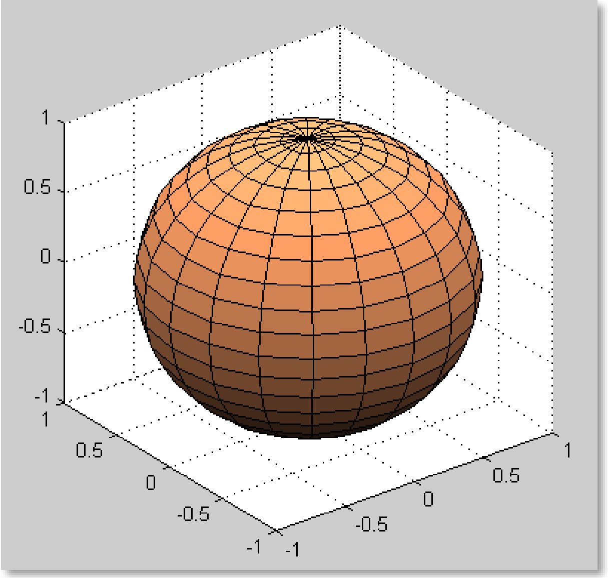
Basis functions



Parametric Surfaces

- Sphere in 3D

$$s : \mathbb{R}^2 \to \mathbb{R}^3$$



$$s(u, v) = r\left(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)\right)$$

$$(u, v) \in [0, 2\pi) \times [-\pi/2, \pi/2]$$

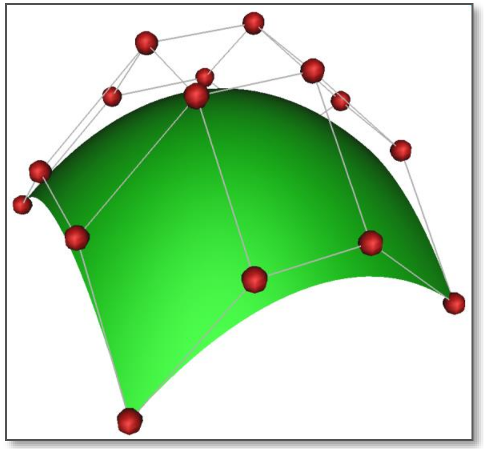
Parametric Surfaces

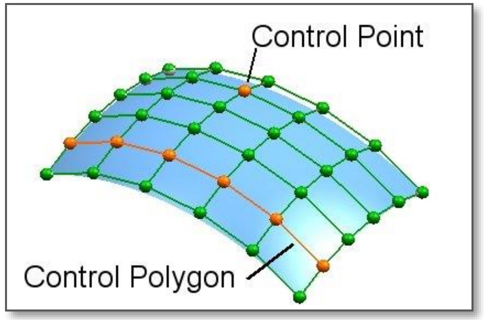
- Curve swept by another curve

$$\mathbf{p}(u,v) = \sum_{i,j} \mathbf{p}_{i,j} B_i(u) B_j(v)$$

- Bézier surface:

$$\mathbf{p}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{p}_{i,j} B_i^m(u) B_j^n(v)$$



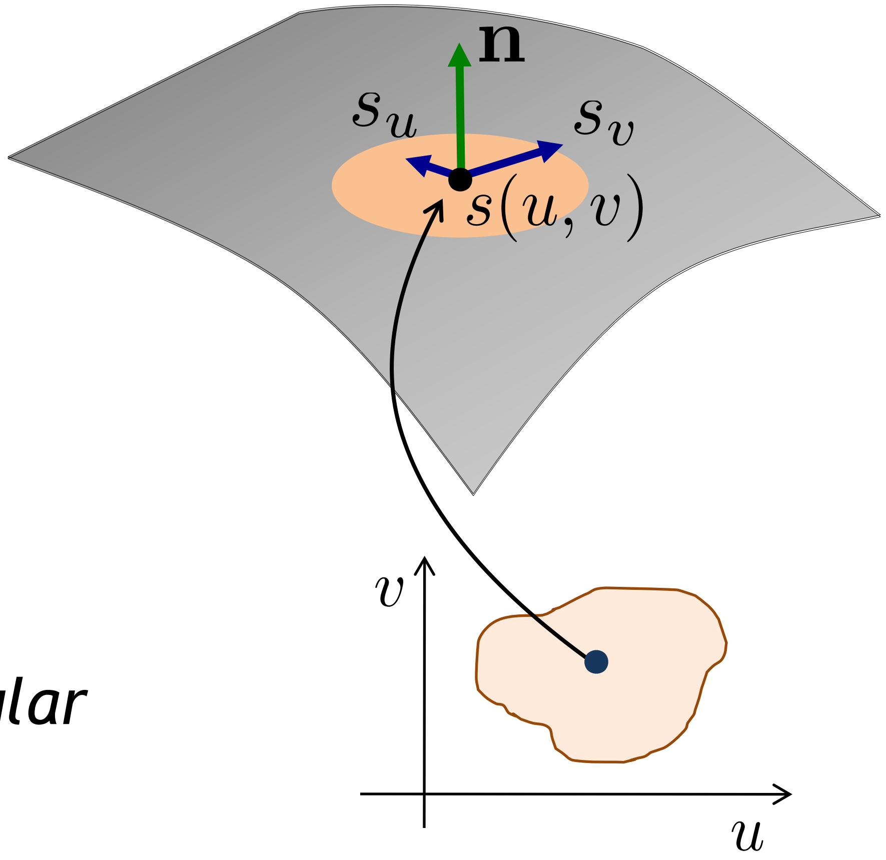


Tangents and Normal

$$s_u = \frac{\partial s(u,v)}{\partial u}$$

$$s_v = \frac{\partial s(u,v)}{\partial v}$$

$$\mathbf{n} = \frac{s_u \times s_v}{\|s_u \times s_v\|}$$



$\mathbf{n}$ exists if parameterization is *regular*

Tangent plane is normal to $\mathbf{n}$

Parametric Curves and Surfaces

- Advantages
  - Easy to generate points on the curve/surface
  - Easy to compute tangents, normal, etc.

- Disadvantages
  - Hard to determine inside/outside
  - Hard to determine if a point is **on** the curve/surface

Implicit Curves and Surfaces

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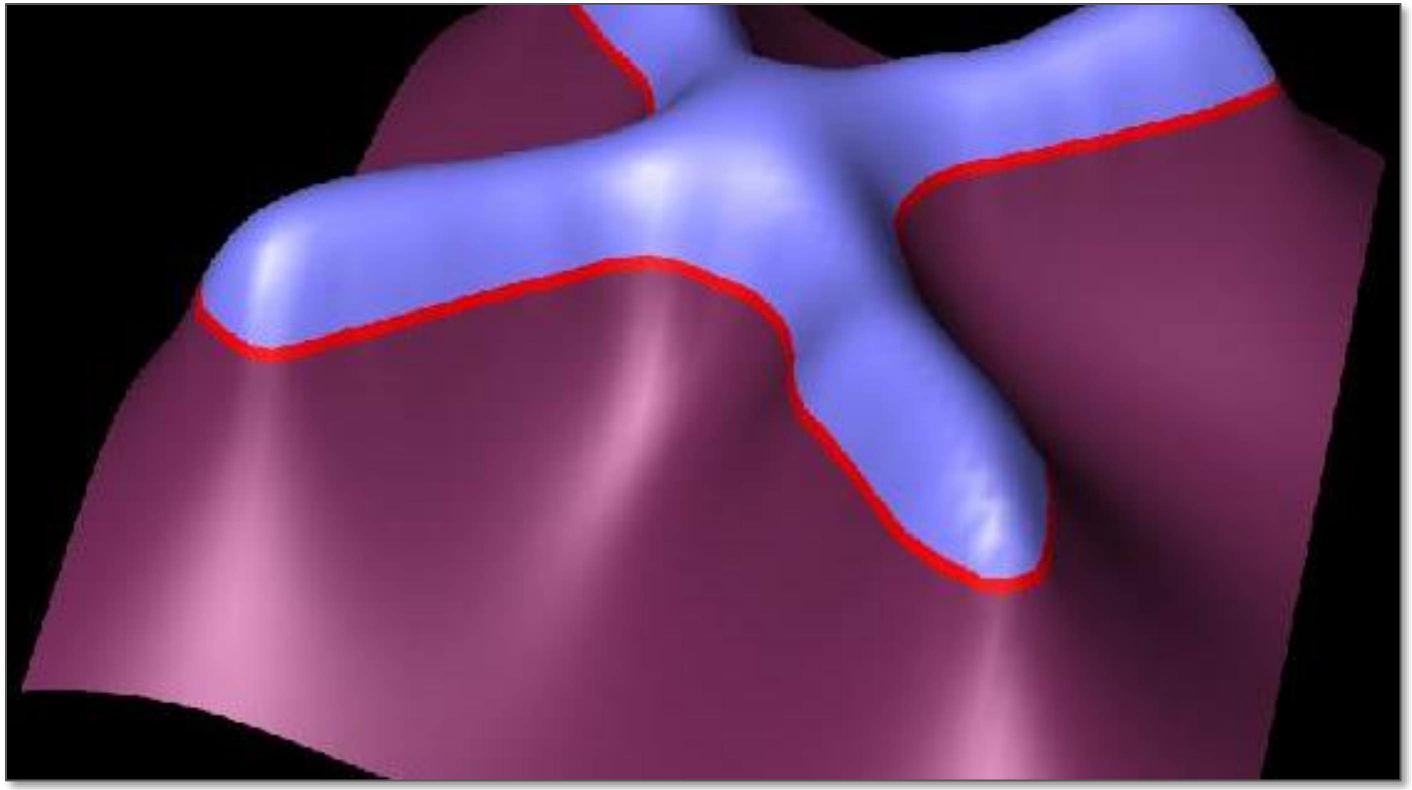
Implicit Curves and Surfaces



Implicit Curves and Surfaces

(usually the zero level set)

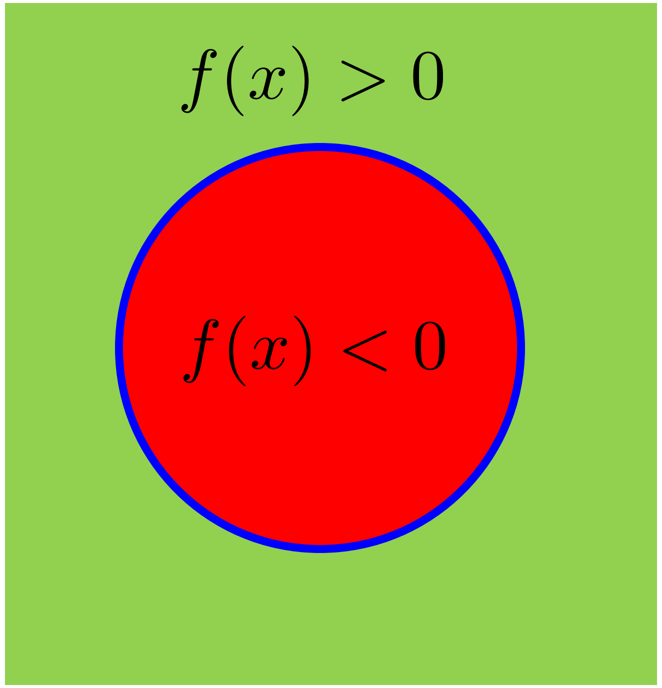
- **Level set of a scalar function** $f : \mathbb{R}^m \to \mathbb{R}$
  - Curve in 2D: $S = \{x \in \mathbb{R}^2 | f(x) = 0\}$
  - Surface in 3D: $S = \{x \in \mathbb{R}^3 | f(x) = 0\}$

- Space partitioning

$\{x \in \mathbb{R}^m | f(x) > 0\}$ Outside

$\{x \in \mathbb{R}^m | f(x) = 0\}$ Curve/Surface

$\{x \in \mathbb{R}^m | f(x) < 0\}$ Inside



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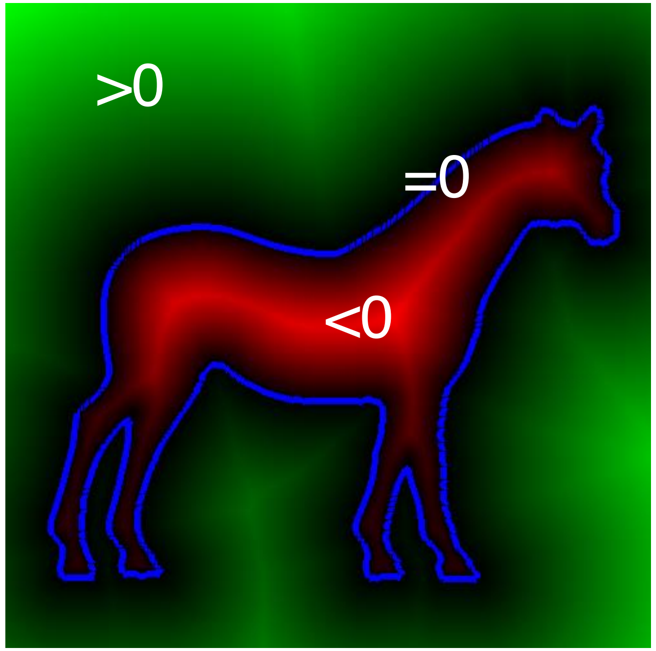
- Popular choice: zero level set of the **signed distance function**



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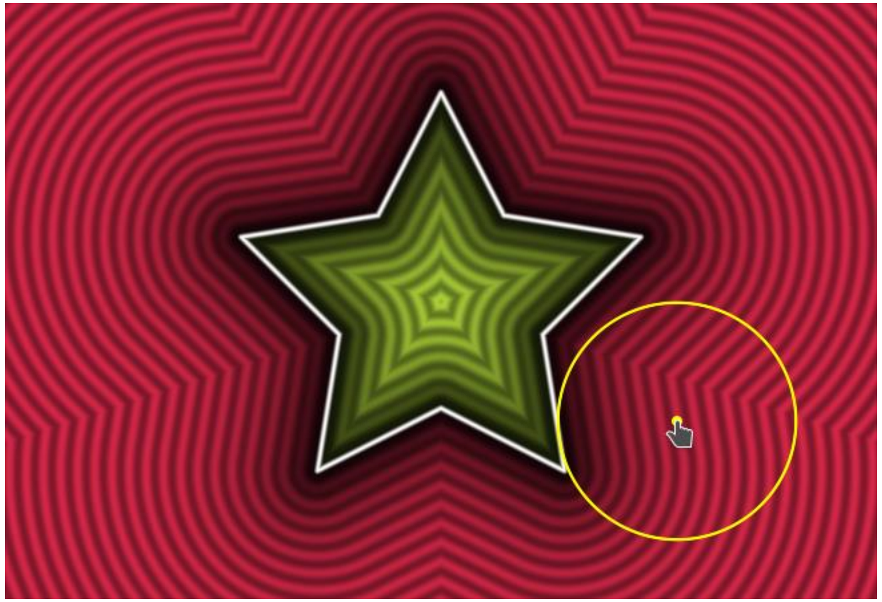
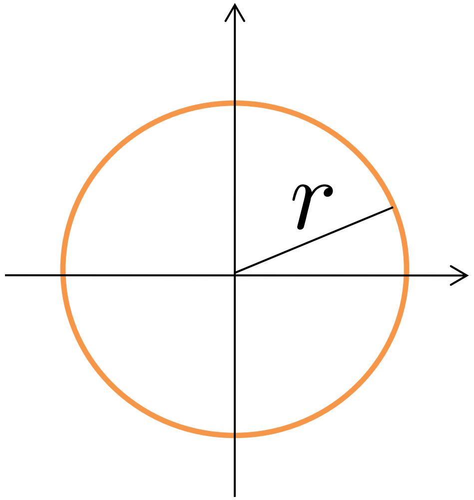


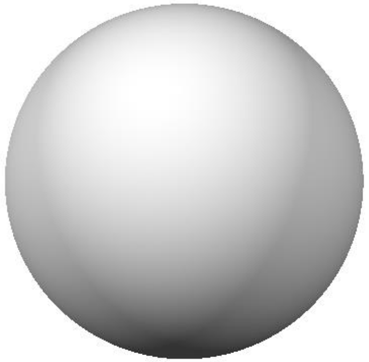
Implicit Curves and Surfaces

- Implicit circle and sphere

$$f(x, y) = x^2 + y^2 - r^2$$



$$f(x, y, z) = x^2 + y^2 + z^2 - r^2$$



… can convert to true signed distance using sqrt

Implicit Curves and Surfaces

- The normal direction to the surface (or curve) is given by the gradient of the implicit function

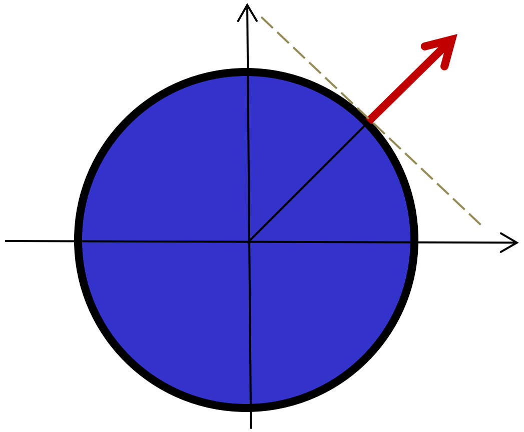
$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^{\mathsf{T}}$$

Why?

- Example

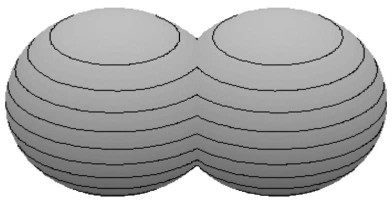
$$f(x,y,z) = x^2 + y^2 + z^2 - r^2$$

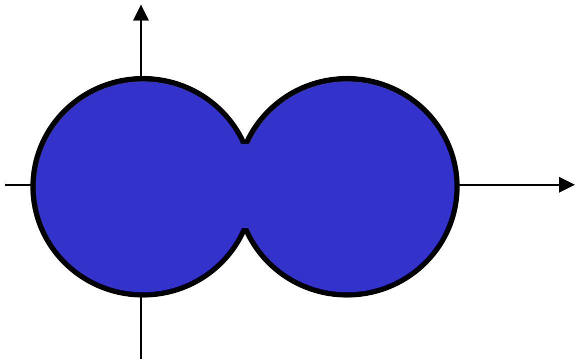
$$\nabla f(x,y,z) = (2x, 2y, 2z)^{\mathsf{T}}$$



Boolean Set Operations

- Union: $$\bigcup_i f_i(x) = \min f_i(x)$$



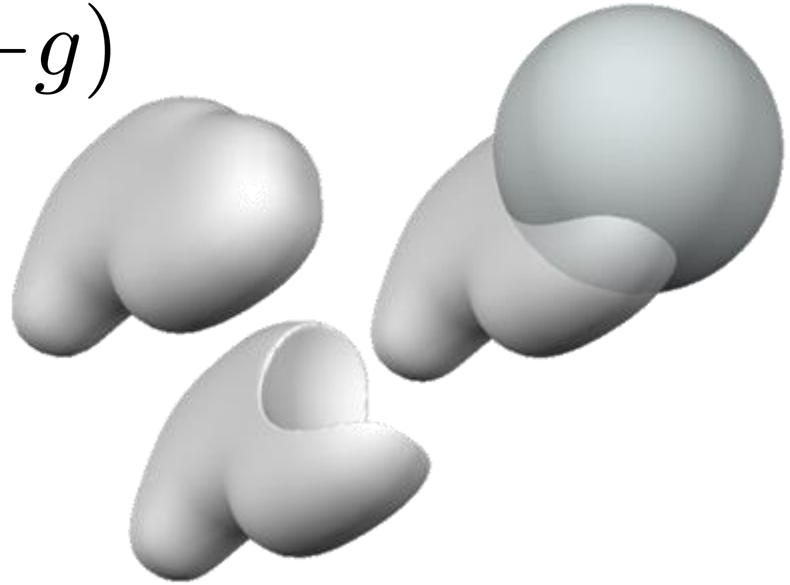


- Intersection: $$\bigcap_i f_i(x) = \max f_i(x)$$

Boolean Set Operations

- Positive = outside, negative = inside
- Boolean subtraction: $h = max(f, -g)$

	$f > 0$	$f < 0$
$g > 0$	$h > 0$	$h < 0$
$g < 0$	$h > 0$	$h > 0$



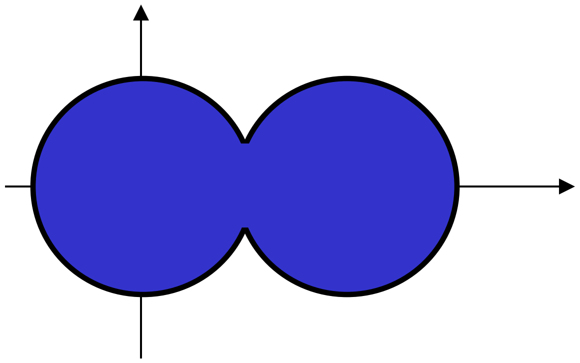
- Much easier than for parametric surfaces!

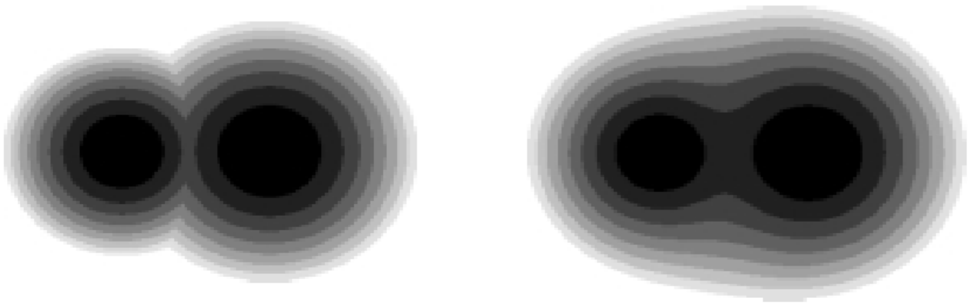
Smooth Set Operations

- In many cases, smooth blending is desired
  - Pasko and Savchenko, “Blending operations for the functionally based constructive geometry” [1994]

$$f \cup g = \frac{1}{1+\alpha}\left(f + g - \sqrt{f^2 + g^2 - 2\alpha f g}\right)$$

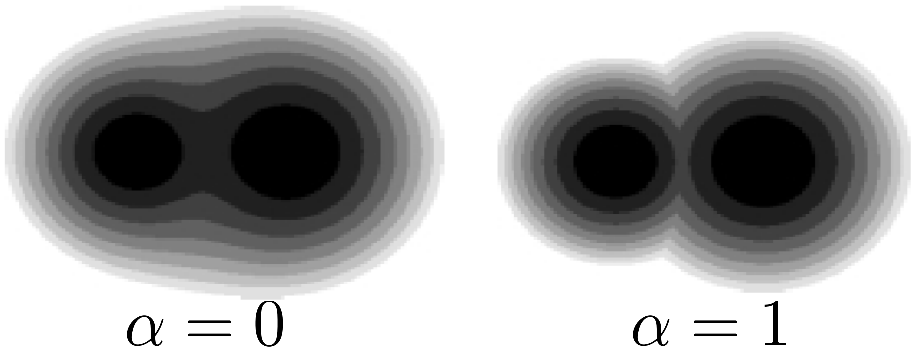
$$f \cap g = \frac{1}{1+\alpha}\left(f + g + \sqrt{f^2 + g^2 - 2\alpha f g}\right)$$





Smooth Set Operations

- Examples



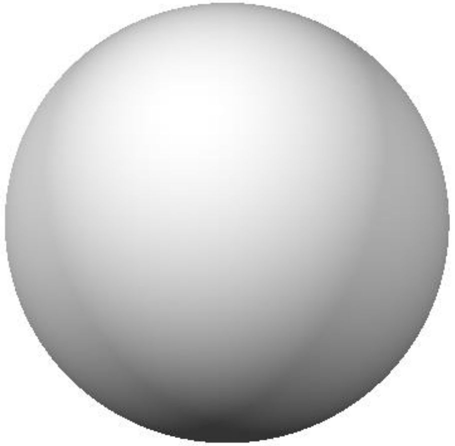
- For $\alpha = 1$, this is equivalent to $\min$ and $\max$

$$\lim_{\alpha \to 1} f \cup g = \frac{1}{2}\left(f + g - \sqrt{(f-g)^2}\right) = \frac{f+g}{2} - \frac{|f-g|}{2} = \min(f,g)$$

$$\lim_{\alpha \to 1} f \cap g = \frac{1}{2}\left(f + g + \sqrt{(f-g)^2}\right) = \frac{f+g}{2} + \frac{|f-g|}{2} = \max(f,g)$$

Designing with Implicit Surfaces

- Sphere: zero level set of this function:



$$f(\mathbf{p}) = \|\mathbf{p}\|^2 - r^2$$

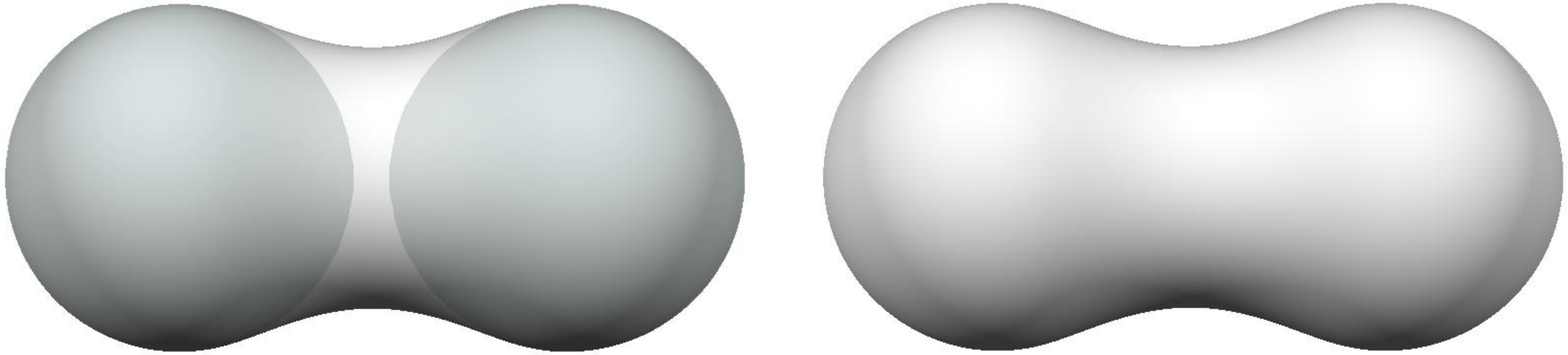
- But also the level set **at value e^{-1}** of this function:

$$f(\mathbf{p}) = e^{-\|\mathbf{p}\|^2/r^2}$$

Designing with Implicit Surfaces

- With smooth falloff functions, adding implicit functions generates a blend:

$$f(\mathbf{p}) = e^{-\|\mathbf{p}-\mathbf{p}_1\|^2} + e^{-\|\mathbf{p}-\mathbf{p}_2\|^2}$$



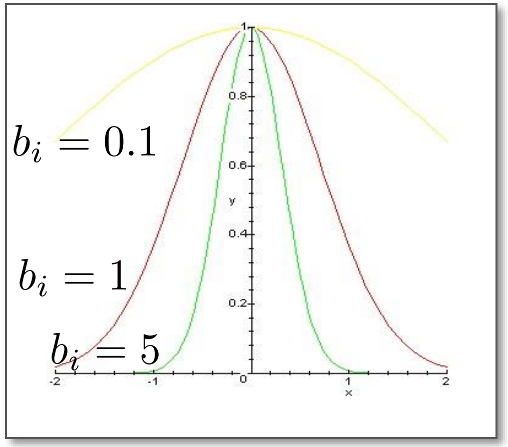
- Called “Metaballs” or “Blobs”

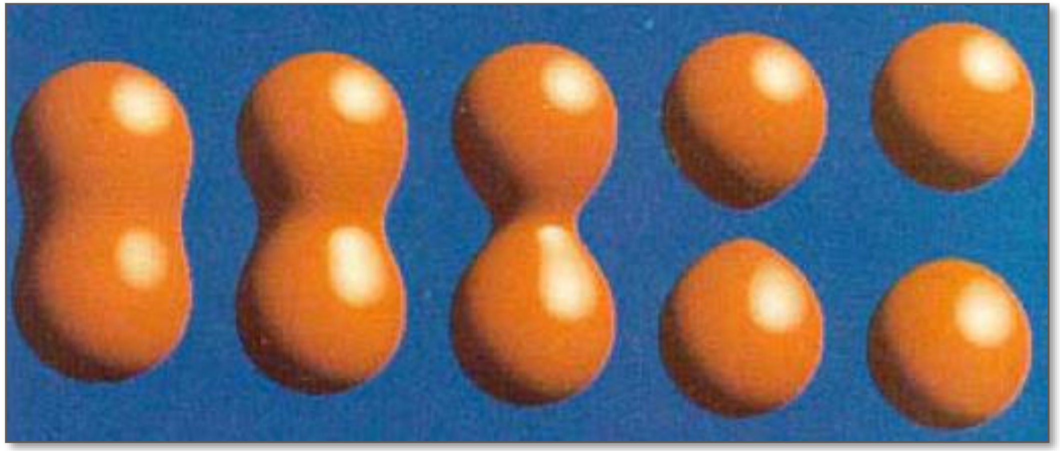
Blobs

- Suggested by Blinn [1982]
  - Defined implicitly by a potential function around a point $\mathbf{p}_i$:

$$f(\mathbf{p}) = a_i e^{-b_i \|\mathbf{p}-\mathbf{p}_i\|^2}$$

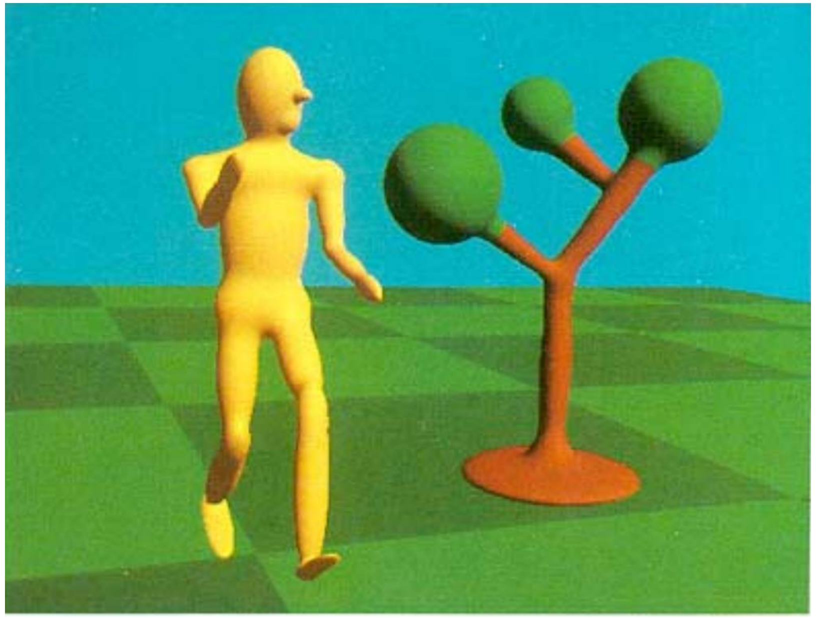
  - Set operations by simple addition/subtraction



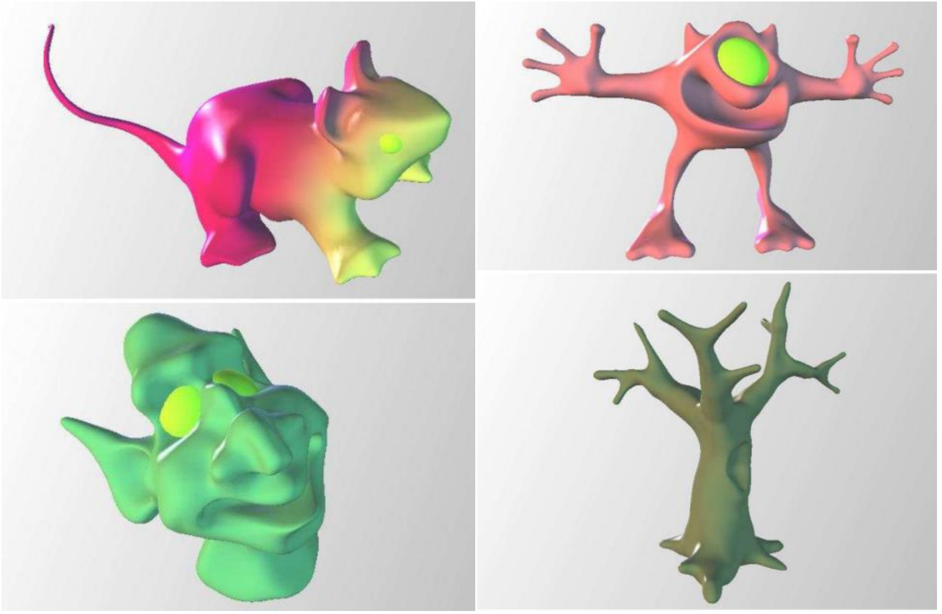


J. Blinn, "A Generalization of Algebraic Surface Drawing", ACM Transactions on Graphics, Vol. 1, No. 3, pp. 235-256, July, 1982.

Blobs



J. Blinn, "A Generalization of Algebraic Surface Drawing", ACM Transactions on Graphics, Vol. 1, No. 3, pp. 235-256, July, 1982.



Angelidis et al., “Swirling-Sweepers: Constant-Volume Modeling”, Pacific Graphics 2004

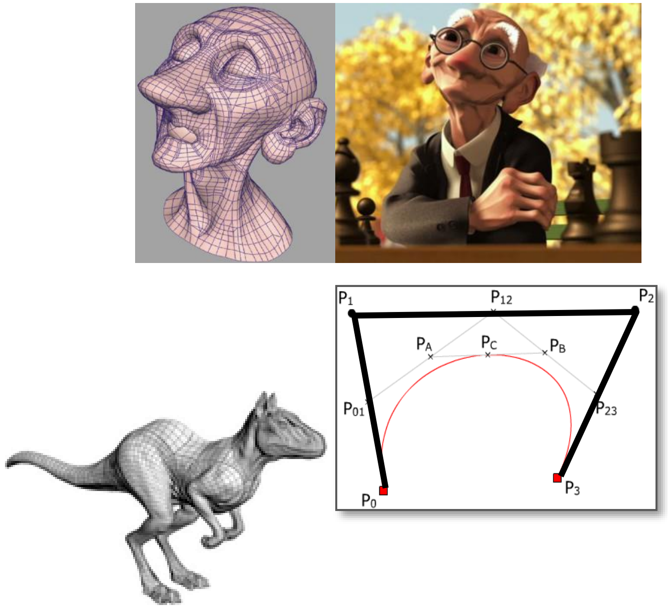
Implicit Curves and Surfaces

- Advantages
  - Easy to determine inside/outside
  - Easy to determine if a point is **on** the curve/surface
  - *Regular* sampling of the entire space, like a grid (good e.g. for neural networks)

- Disadvantages
  - Hard to generate points on the curve/surface
  - Does not easily lend itself to (real-time) rendering

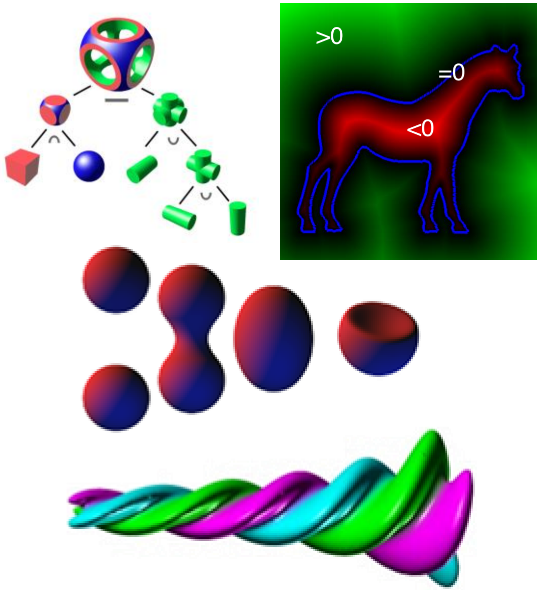
Summary

Parametric



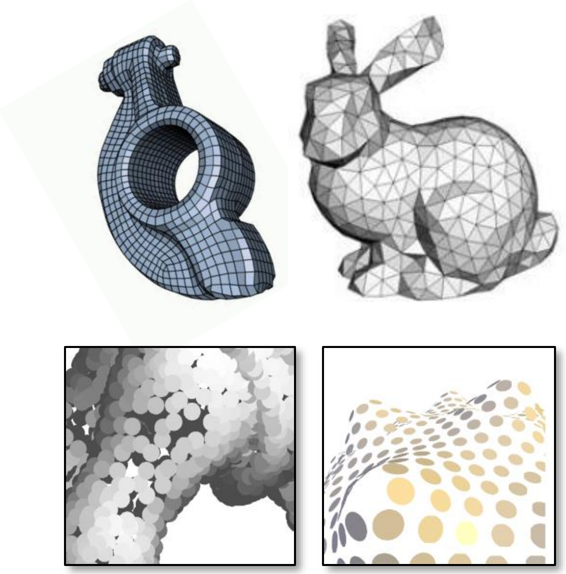
- Splines, tensor-product surfaces
- Subdivision surfaces

Implicit



- Metaballs/blobs
- Distance fields

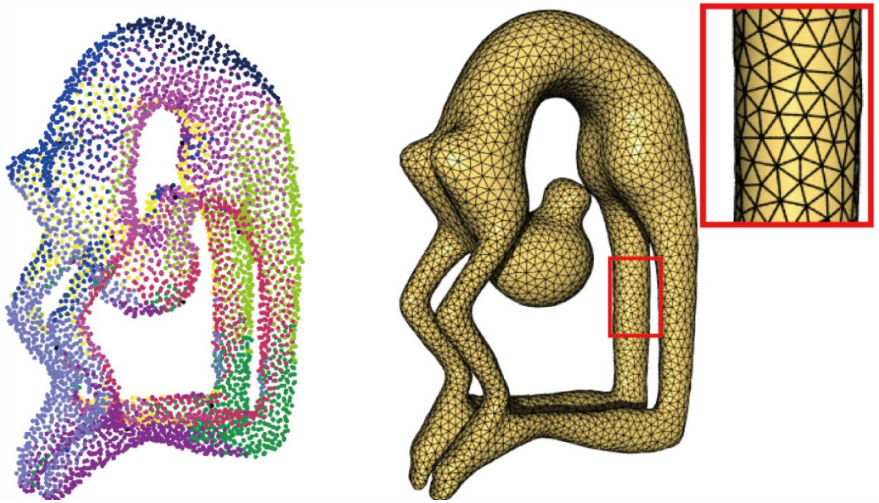
Discrete/Sampled



- Meshes
- Point set surfaces

In the Next Lectures

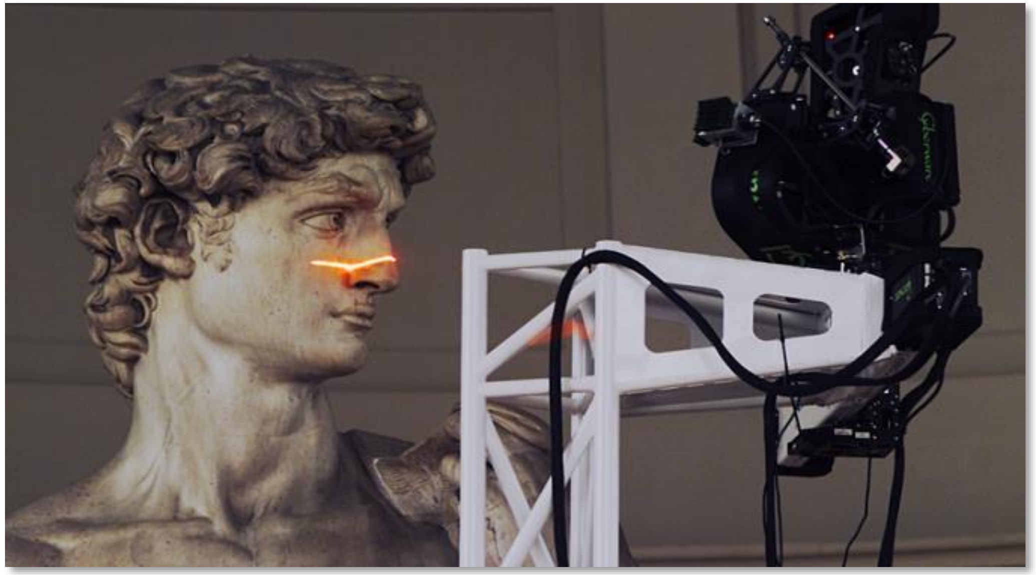
- How to get a clean, watertight surface mesh from a sampled point set

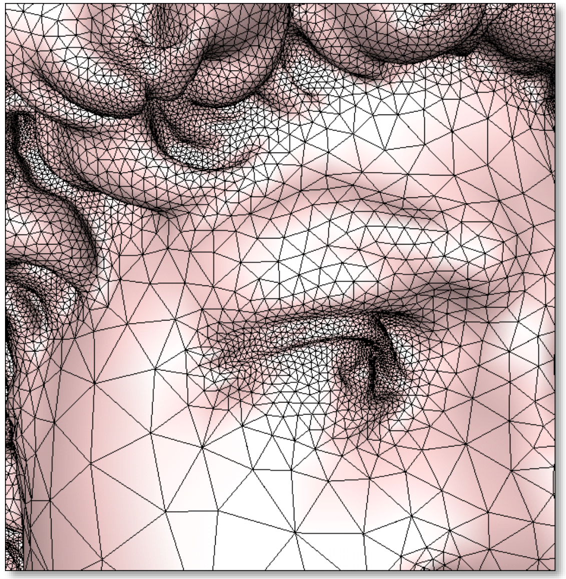


- The most popular way:
points ➐implicit function ➐surface mesh

Next Week

- A bit about geometry acquisition
- All About Meshes





Thank you