

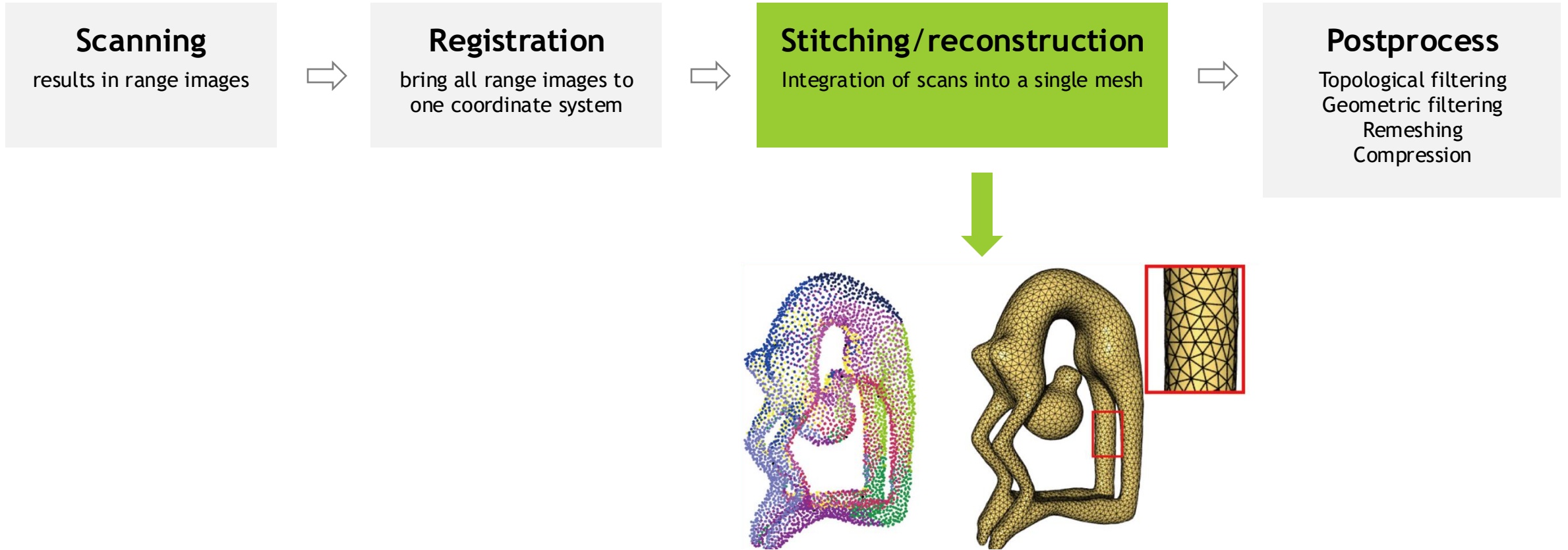
252-0538-00L, Spring 2025

# Shape Modeling and Geometry Processing

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## Surface Reconstruction

# Geometry Acquisition Pipeline



# Digital Michelangelo Project



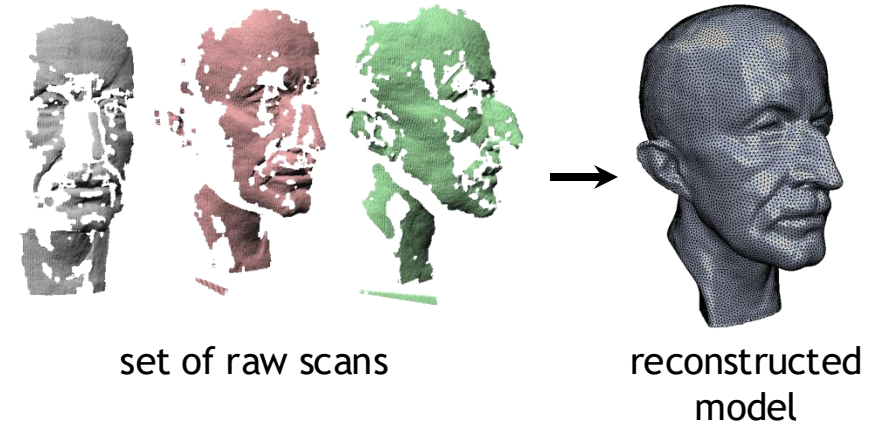
1G sample points → 8M triangles



4G sample points → 8M triangles

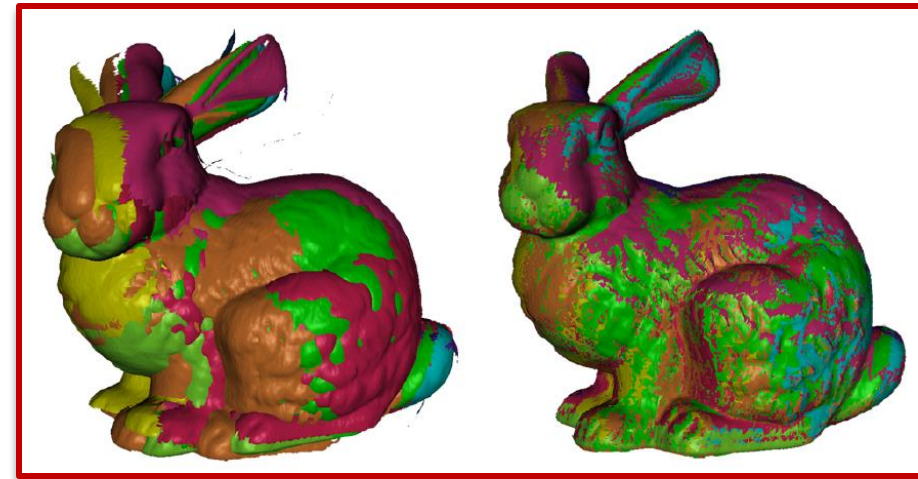
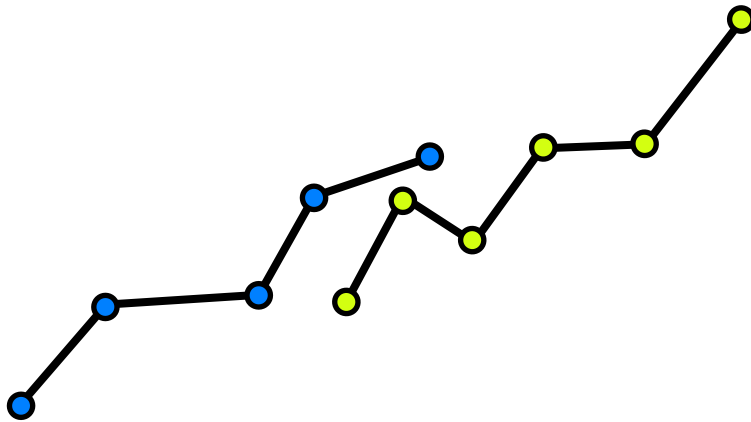
# Input to Reconstruction Process

- **Input option 1**  
Just a set of 3D points, irregularly spaced
  - Need to estimate normals  
→ next class
- **Input option 2**  
Normals come from the range scans



# How to Connect the Dots?

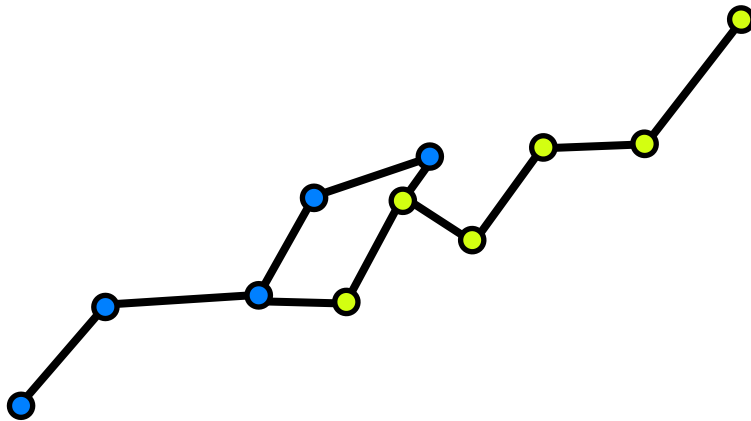
- **Explicit reconstruction:**  
stitch the range scans together



“Zippered Polygon Meshes from Range Images”, Greg Turk and Marc Levoy, ACM SIGGRAPH 1994

# How to Connect the Dots?

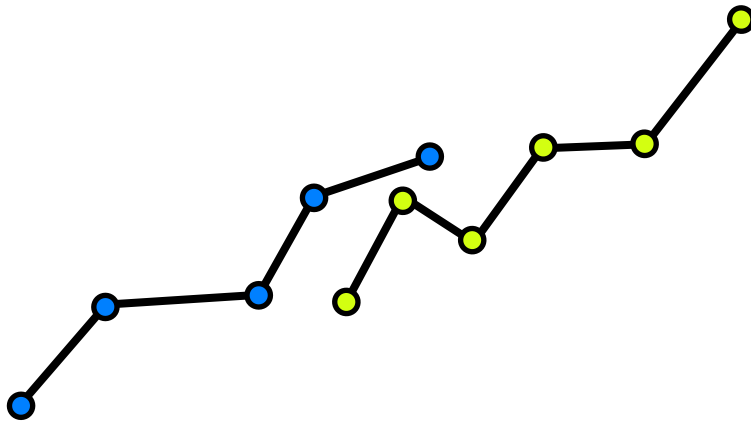
- **Explicit reconstruction:**  
stitch the range scans together



- Connect sample points by triangles
- Exact interpolation of sample points
- Bad for noisy or misaligned data
- Can lead to holes or non-manifold situations

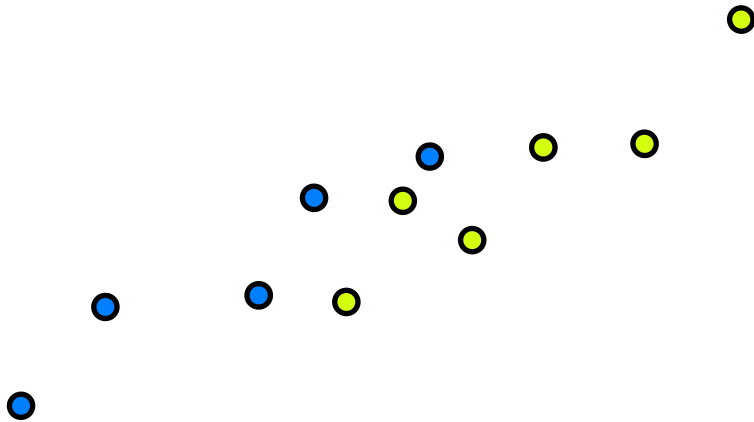
# How to Connect the Dots?

- **Implicit reconstruction:** estimate a signed distance function (SDF); extract 0-level set mesh using Marching Cubes



# How to Connect the Dots?

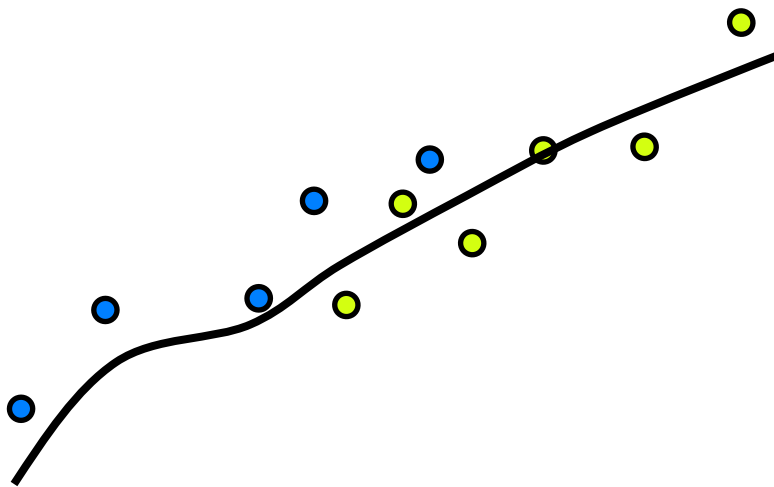
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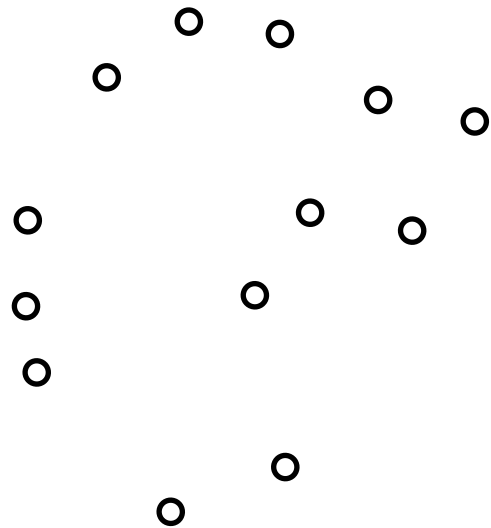
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- Approximation of input points
- Watertight manifold results by construction

# How to Connect the Dots?

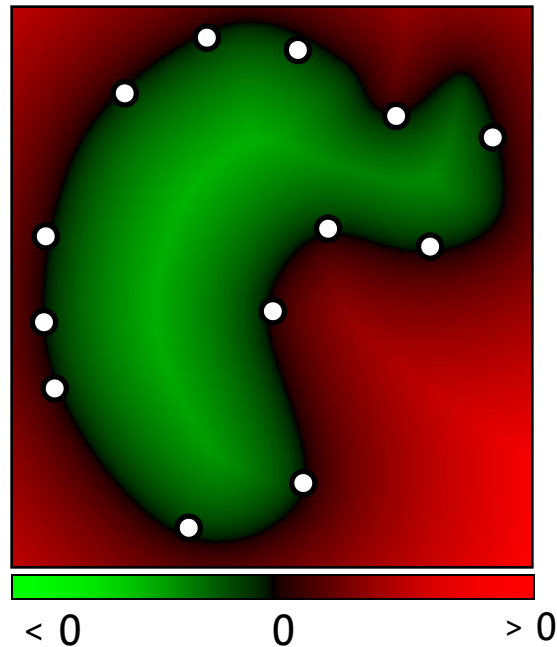
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- Approximation of input points
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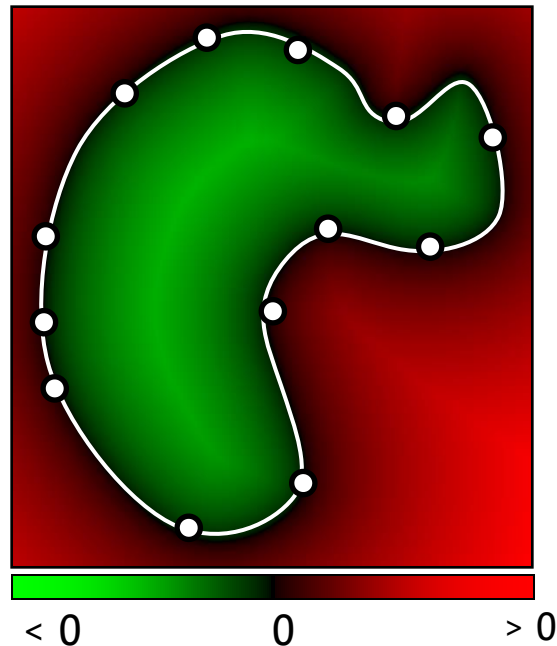
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- Approximation of input points
- Watertight manifold results by construction

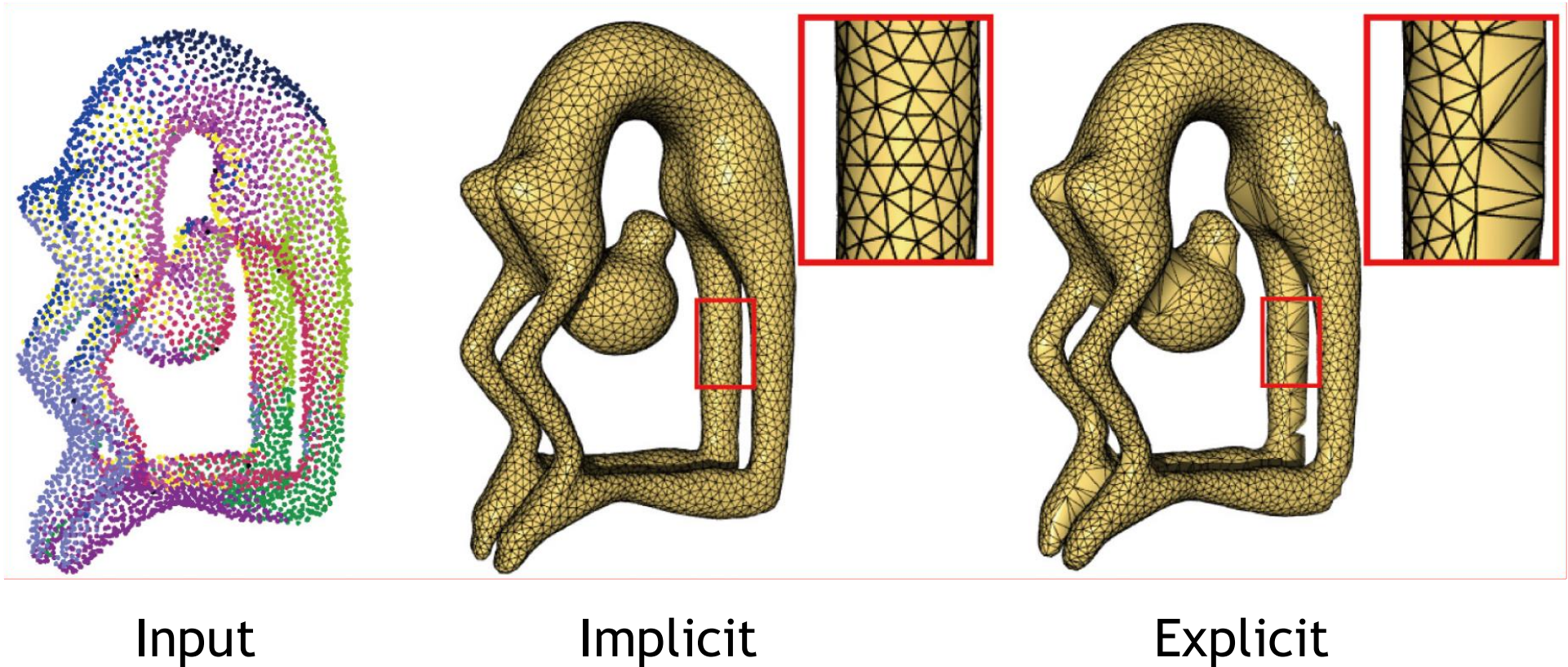
# How to Connect the Dots?

- **Implicit reconstruction:** estimate a signed distance function (SDF); extract 0-level set mesh using Marching Cubes



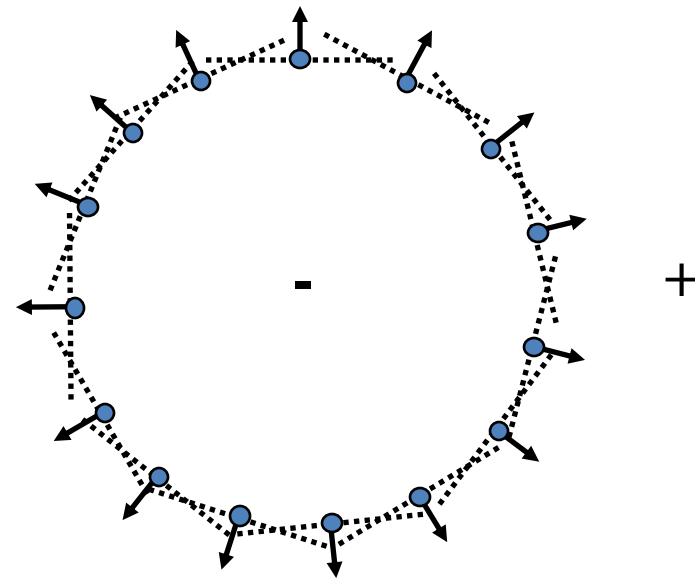
- Approximation of input points
- Watertight manifold results by construction

# Implicit vs. Explicit



# SDF from Points and Normals

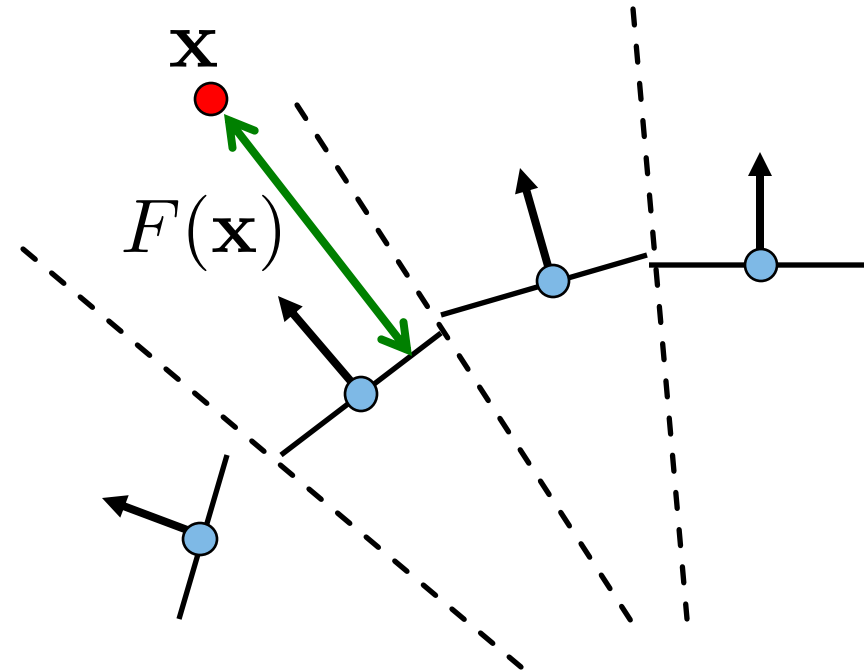
- Compute signed distance to the tangent plane of the closest point
- Normals help to distinguish between inside and outside



“Surface reconstruction from unorganized points”, Hoppe et al., ACM SIGGRAPH 1992  
<http://research.microsoft.com/en-us/um/people/hoppe/proj/recon/>

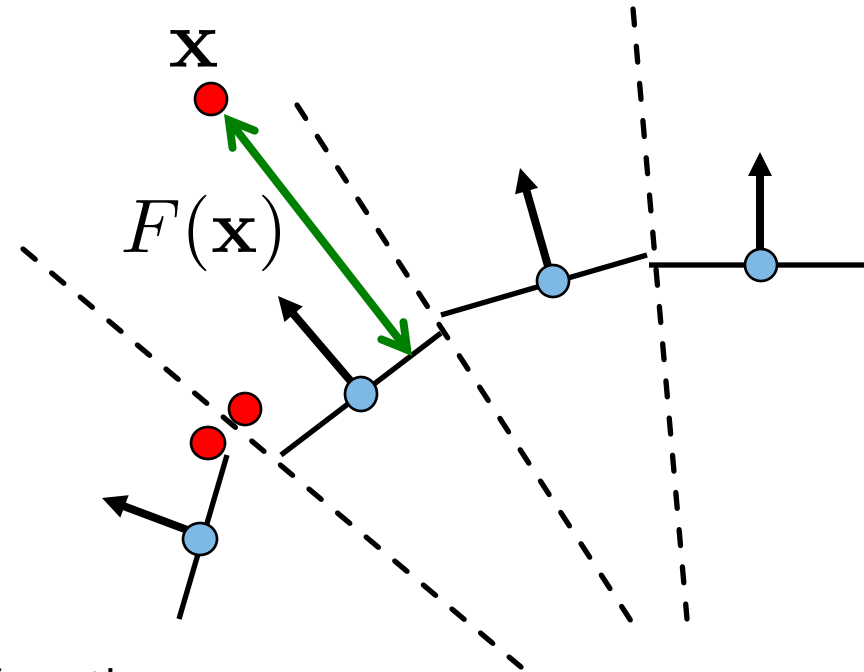
# SDF from Points and Normals

- Compute signed distance to the tangent plane of the closest point
- Problem??



# SDF from Points and Normals

- Compute signed distance to the tangent plane\* of the closest point
- The function will be discontinuous

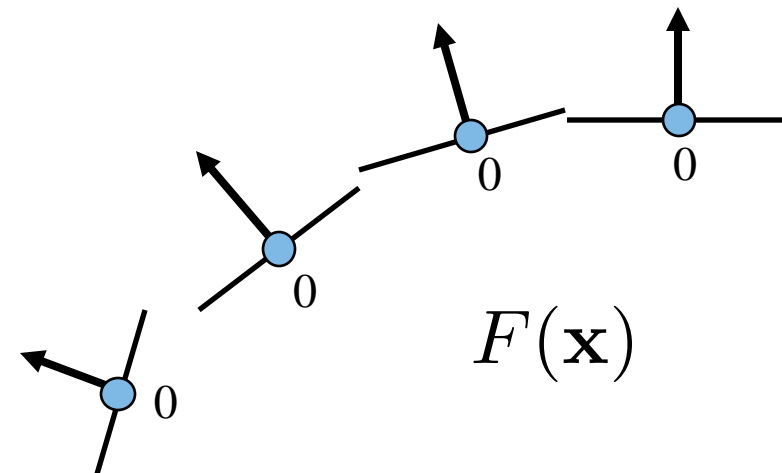


\* The Hoppe92 paper computes the tangent planes slightly differently (by PCA on k-nearest-neighbors of each data point, see next class), but the consequences are still the same.



# Smooth SDF

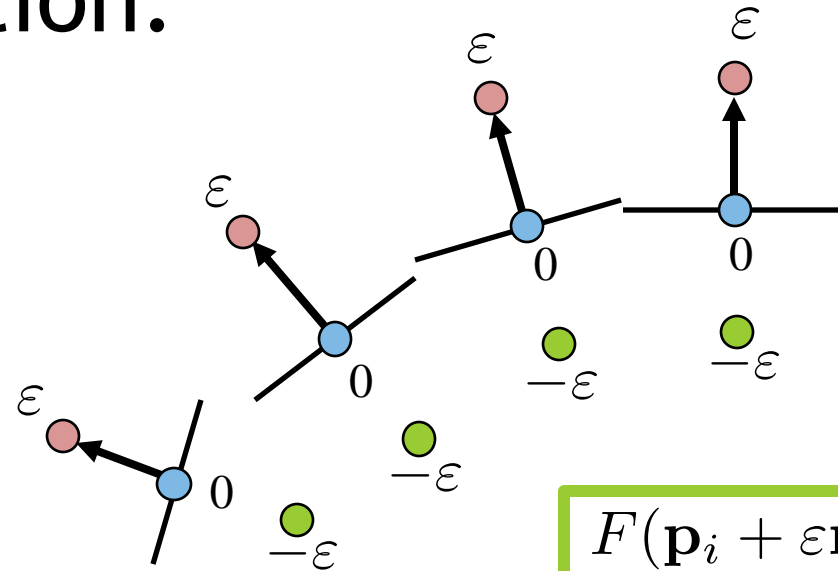
- Instead find a smooth formulation for  $F$ .
- Scattered data interpolation:
  - $F(\mathbf{p}_i) = 0$
  - $F$  is smooth
  - Avoid trivial  $F \equiv 0$



“Reconstruction and representation of 3D objects with radial basis functions”, Carr et al., ACM SIGGRAPH 2001

# Smooth SDF

- Instead find a smooth formulation for  $F$ .
- Scattered data interpolation:
  - $F(\mathbf{p}_i) = 0$
  - $F$  is smooth
  - Avoid trivial  $F \equiv 0$
- Add off-surface constraints



$$\begin{aligned} F(\mathbf{p}_i + \epsilon \mathbf{n}_i) &= \epsilon \\ F(\mathbf{p}_i - \epsilon \mathbf{n}_i) &= -\epsilon \end{aligned}$$

# Radial Basis Function Interpolation

- **RBF:** Weighted sum of shifted, smooth kernels

$$F(\mathbf{x}) = \sum_{m=0}^{N-1} w_m \varphi(\|\mathbf{x} - \mathbf{c}_m\|) + p(\mathbf{x})$$

Scalar coefficients  
**Unknowns**

$$N = 3n$$

Smooth kernels  
(**basis functions**)  
centered at constrained points.  
For example:  
 $\varphi(r) = r^3$

Linear polynomial with  
**unknown** coeffs  
 $p(\mathbf{x}) = a_0 + a_1x + a_2y + a_3z$

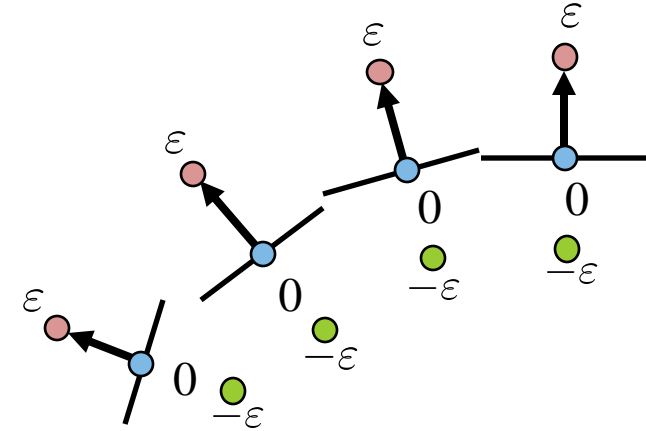
# Radial Basis Function Interpolation

- Interpolate the constraints:

$$F(\mathbf{p}_i) = 0$$

$$F(\mathbf{p}_i + \varepsilon \mathbf{n}_i) = \varepsilon$$

$$F(\mathbf{p}_i - \varepsilon \mathbf{n}_i) = -\varepsilon$$



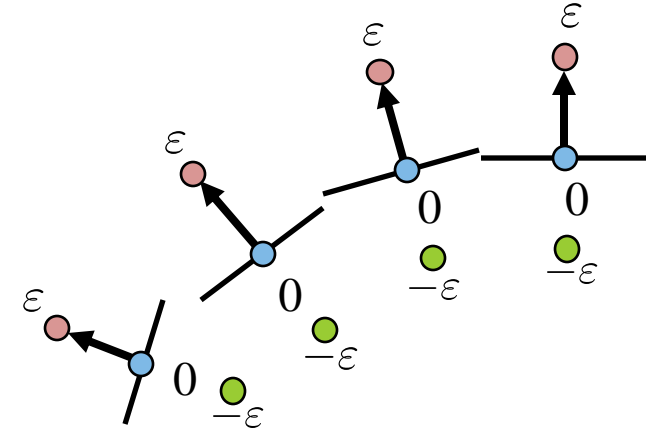
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$$\{\mathbf{c}_{3i}, \mathbf{c}_{3i+1}, \mathbf{c}_{3i+2}\} := \{\mathbf{p}_i, \mathbf{p}_i + \varepsilon \mathbf{n}_i, \mathbf{p}_i - \varepsilon \mathbf{n}_i\}$$

$$d_m = \begin{cases} 0 & m = 3i \\ \varepsilon & m = 3i + 1 \\ -\varepsilon & m = 3i + 2 \end{cases}$$

➡  $F(\mathbf{c}_m) = d_m, \quad m = 0, \dots, N - 1$

# Radial Basis Function Interpolation

- Symmetric linear system to get the coeffs  $w_*$  and  $a_*$ :

$$A = \begin{pmatrix} \varphi(\|\mathbf{c}_0 - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_0 - \mathbf{c}_{N-1}\|) \\ \vdots & \ddots & \vdots \\ \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_{N-1}\|) \end{pmatrix};$$

$$P \in \mathbb{R}^{N \times 4}, \text{ row } m \text{ of } P = (1, c_{x,m}, c_{y,m}, c_{z,m})$$

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} d_0 \\ \vdots \\ d_{N-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- System size  $(N+4) \times (N+4)$
- Dense or sparse, depending on the kernel

# Radial Basis Function Interpolation

$$A = \begin{pmatrix} \varphi(\|\mathbf{c}_0 - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_0 - \mathbf{c}_{N-1}\|) \\ \vdots & \ddots & \vdots \\ \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_{N-1}\|) \end{pmatrix};$$

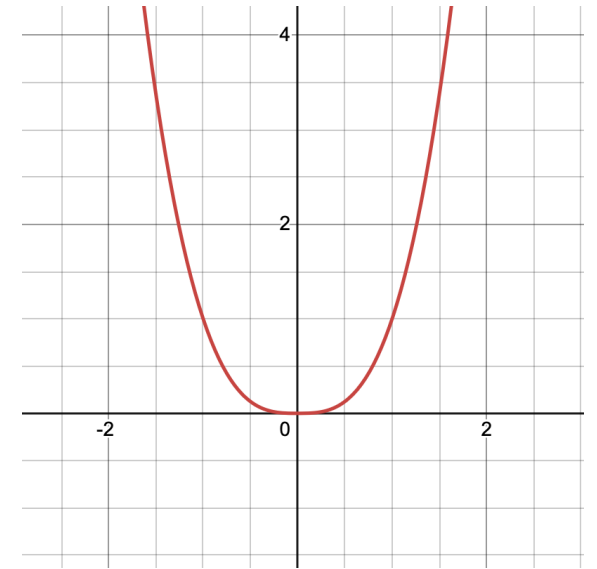
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$$F(\mathbf{x}) = \sum_{m=0}^{N-1} w_m \varphi(\|\mathbf{x} - \mathbf{c}_m\|) + p(\mathbf{x})$$

# RBF Kernels

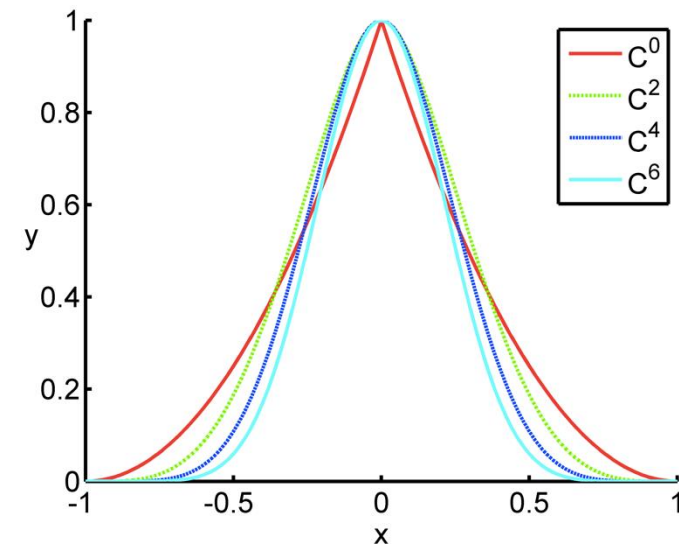
- Triharmonic:  $\varphi(r) = r^3$ 
  - Globally supported
  - Leads to dense symmetric linear system
  - $C^2$  smoothness
  - Works well for highly irregular sampling



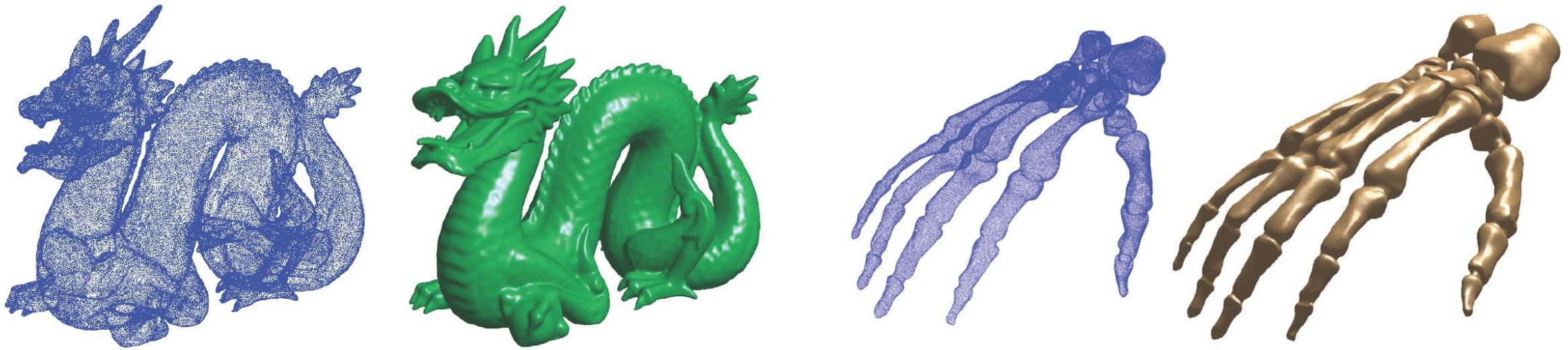


# RBF Kernels

- Polyharmonic
  - $\varphi(r) = r^k \log(r)$ ,  $k = 2, 4, 6 \dots$
  - $\varphi(r) = r^k$ ,  $k = 1, 3, 5 \dots$
- Multiquadratic
$$\varphi(r) = \sqrt{r^2 + \beta^2}$$
- Gaussian
$$\varphi(r) = e^{-\beta r^2}$$
- B-Spline (compact support)
$$\varphi(r) = \text{piecewise-polynomial}(r)$$

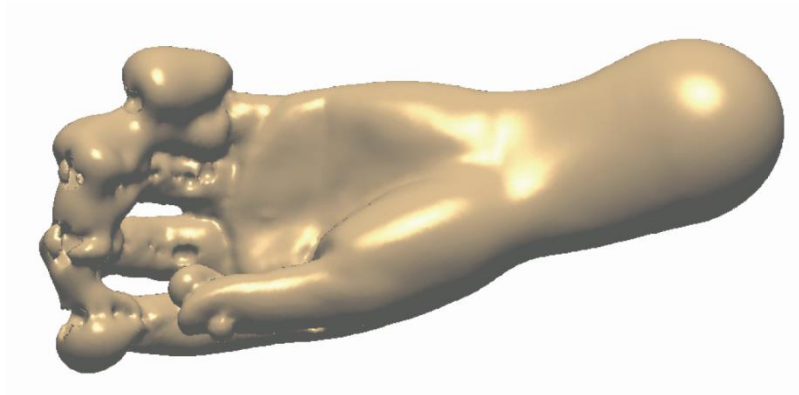


# RBF Reconstruction Examples

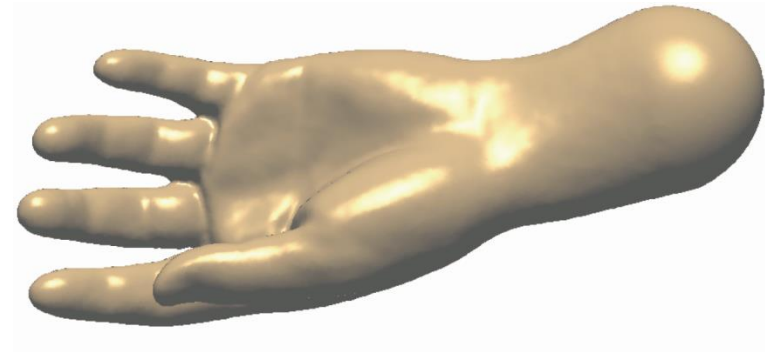


“Reconstruction and representation of 3D objects with radial basis functions”, Carr et al., ACM SIGGRAPH 2001

# Off-Surface Points



Insufficient number/  
badly placed off-surface points



Properly chosen off-surface points

“Reconstruction and representation of 3D objects with radial basis functions”, Carr et al., ACM SIGGRAPH 2001

# Comparison of the various SDFs so far

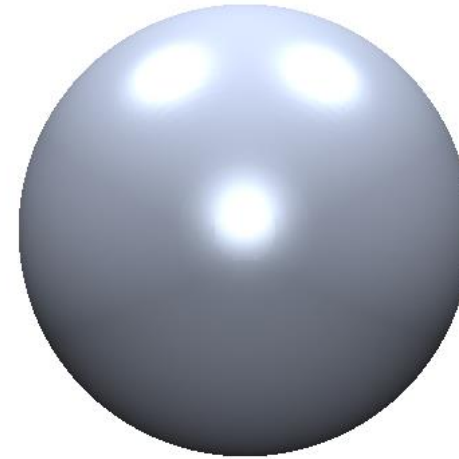
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Distance  
to plane



Compact RBF

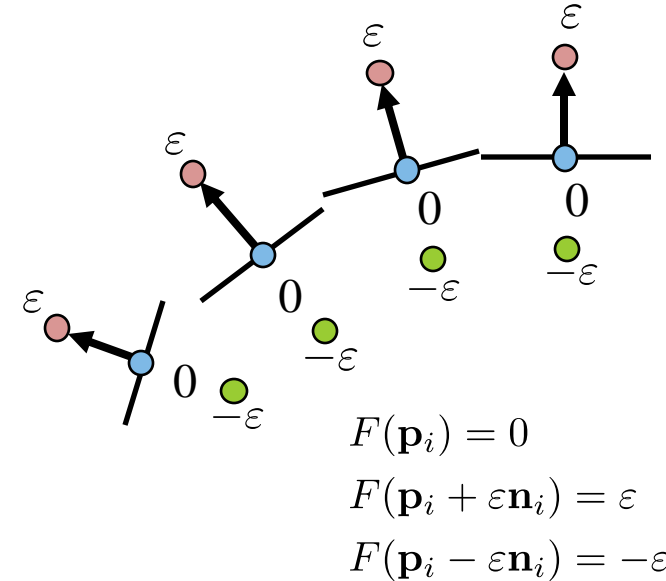


Global RBF  
Triharmonic

# RBF - Discussion

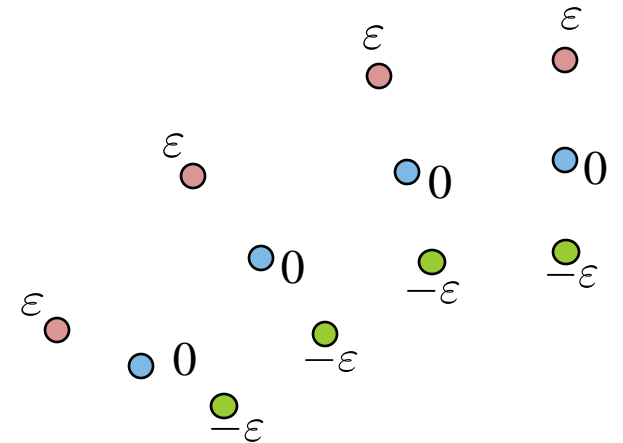
- Global definition!
- Global optimization of the weights, even if the basis functions are local

$$F(\mathbf{x}) = \sum_{m=0}^{N-1} w_m \varphi(\|\mathbf{x} - \mathbf{c}_m\|) + p(\mathbf{x})$$



# Moving Least Squares (MLS)

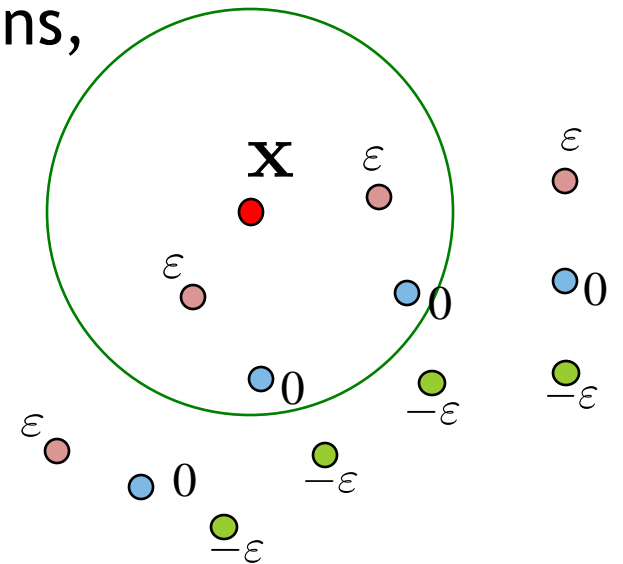
- Do purely **local** approximation of the SDF
- Weights change depending on where we are evaluating
- The beauty: the “stitching” of all local approximations, seen as one function  $F(\mathbf{x})$ , is smooth everywhere!
  - We get a **globally** smooth function but only do **local** computation



“Interpolating and Approximating Implicit Surfaces from Polygon Soup”, Shen et al.,  
ACM SIGGRAPH 2004  
<http://graphics.berkeley.edu/papers/Shen-IAI-2004-08/index.html>

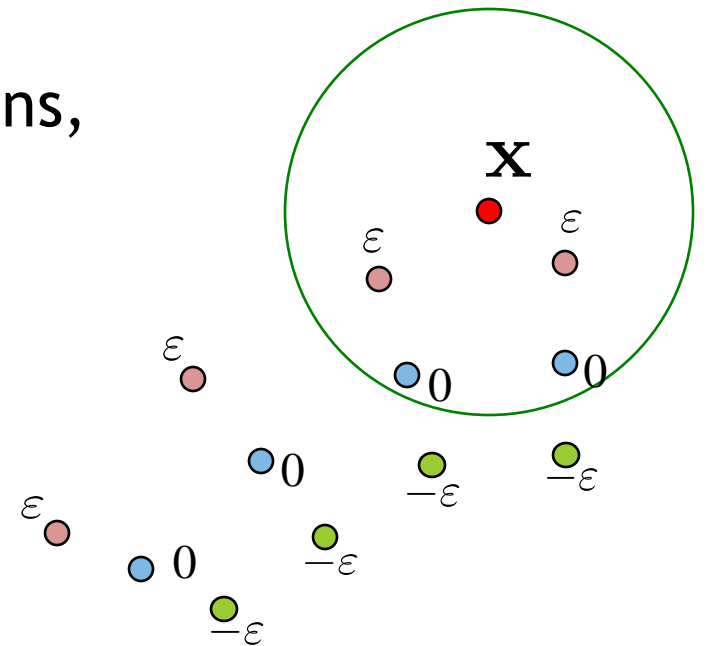
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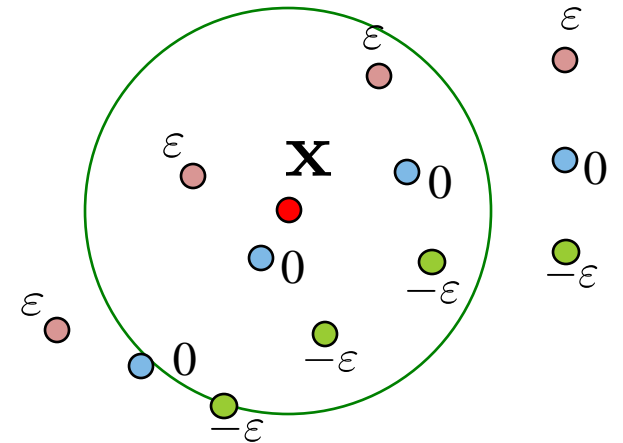
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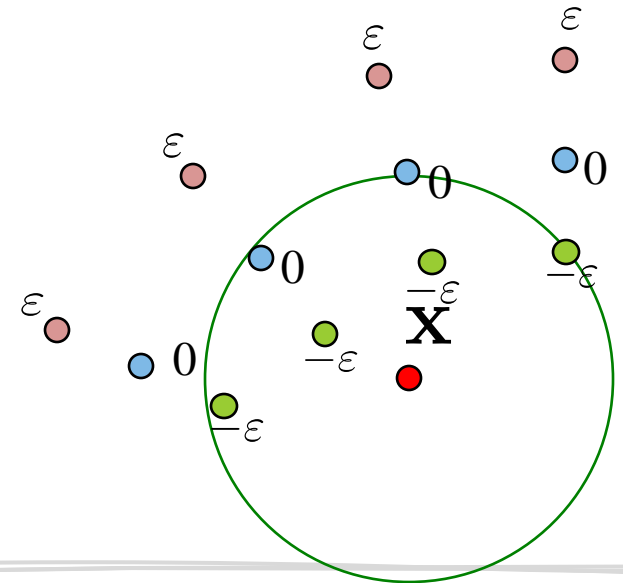
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# Least-Squares Approximation

- Polynomial least-squares approximation
  - Choose degree,  $k$

$$f \in \Pi_k^3 : f(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + \dots + a_*z^k$$

$$f(\mathbf{x}) = \mathbf{b}(\mathbf{x})^\top \mathbf{a}$$

$$\mathbf{a} = (a_1, a_2, \dots, a_*)^\top, \quad \mathbf{b}(\mathbf{x})^\top = (1, x, y, z, x^2, xy, \dots, z^k)$$

- Find  $\mathbf{a}$  that minimizes sum of squared differences

$$\operatorname{argmin}_{f \in \Pi_k^3} \sum_{m=0}^{N-1} (f(\mathbf{c}_m) - d_m)^2 \quad \text{or:} \quad \operatorname{argmin}_{\mathbf{a}} \sum_{m=0}^{N-1} (\mathbf{b}(\mathbf{c}_m)^\top \mathbf{a} - d_m)^2$$

# MOVING Least-Squares Approximation

- Polynomial least-squares approximation
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- Find  $\mathbf{a}_{\mathbf{x}}$  that minimizes WEIGHTED sum of squared differences

$$f_{\mathbf{x}} = \operatorname{argmin}_{f \in \Pi_k^3} \sum_{m=0}^{N-1} \theta(\|\mathbf{x} - \mathbf{c}_m\|) (f(\mathbf{c}_m) - d_m)^2 \quad \text{or:} \quad \mathbf{a}_{\mathbf{x}} = \operatorname{argmin}_{\mathbf{a}} \sum_{m=0}^{N-1} \theta(\|\mathbf{x} - \mathbf{c}_m\|) (\mathbf{b}(\mathbf{c}_m)^\top \mathbf{a} - d_m)^2$$

# MOVING Least-Squares Approximation

- Polynomial least-squares approximation

- Choose degree,  $k$

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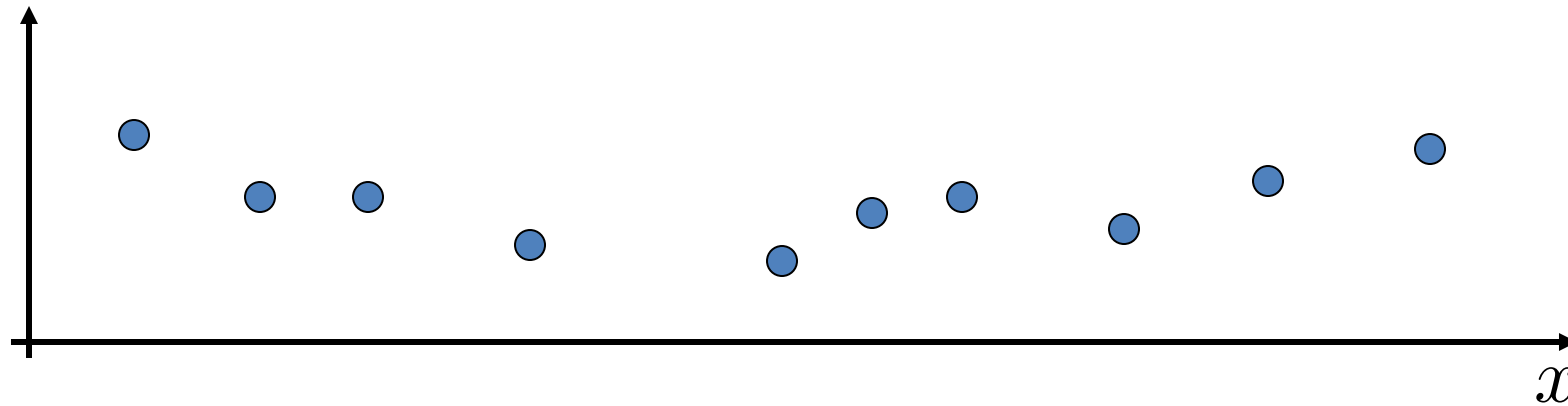
$$\mathbf{a} = (a_1, a_2, \dots, a_*)^\top, \quad \mathbf{b}(\mathbf{x})^\top = (1, x, y, z, x^2, xy, \dots, z^k)$$

- Find  $\mathbf{a}_{\mathbf{x}}$  that minimizes WEIGHTED sum of squared differences
- The value of the SDF is the obtained approximation evaluated at  $\mathbf{x}$ :

$$F(\mathbf{x}) = f_{\mathbf{x}}(\mathbf{x}) = \mathbf{b}(\mathbf{x})^\top \mathbf{a}_{\mathbf{x}}$$

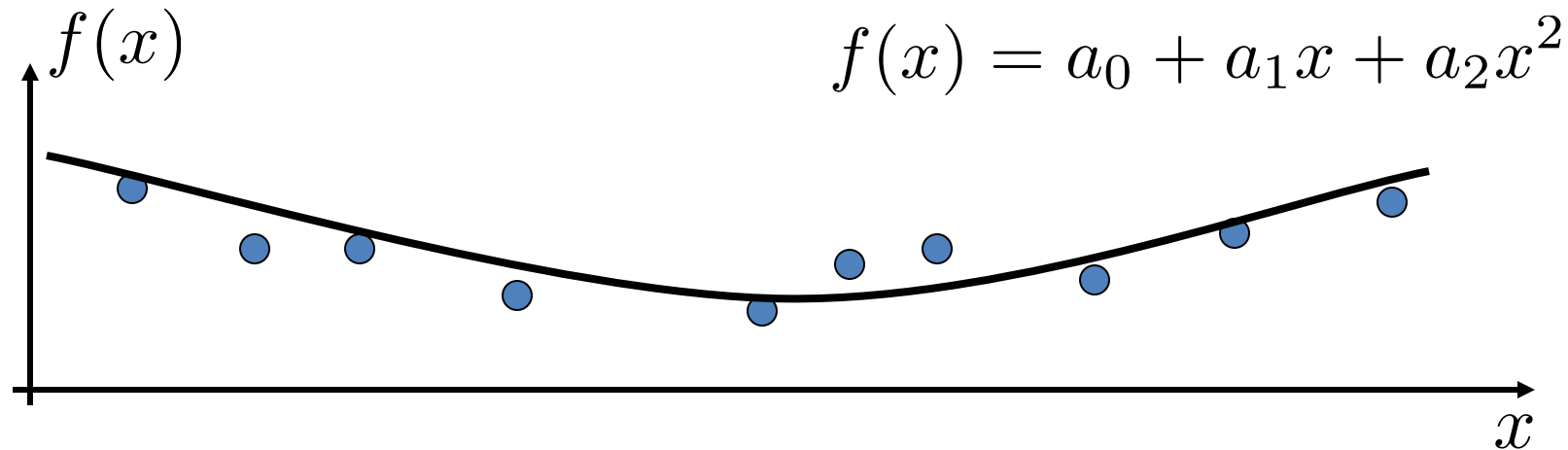
# MLS - 1D Example

- Input values



# MLS - 1D Example

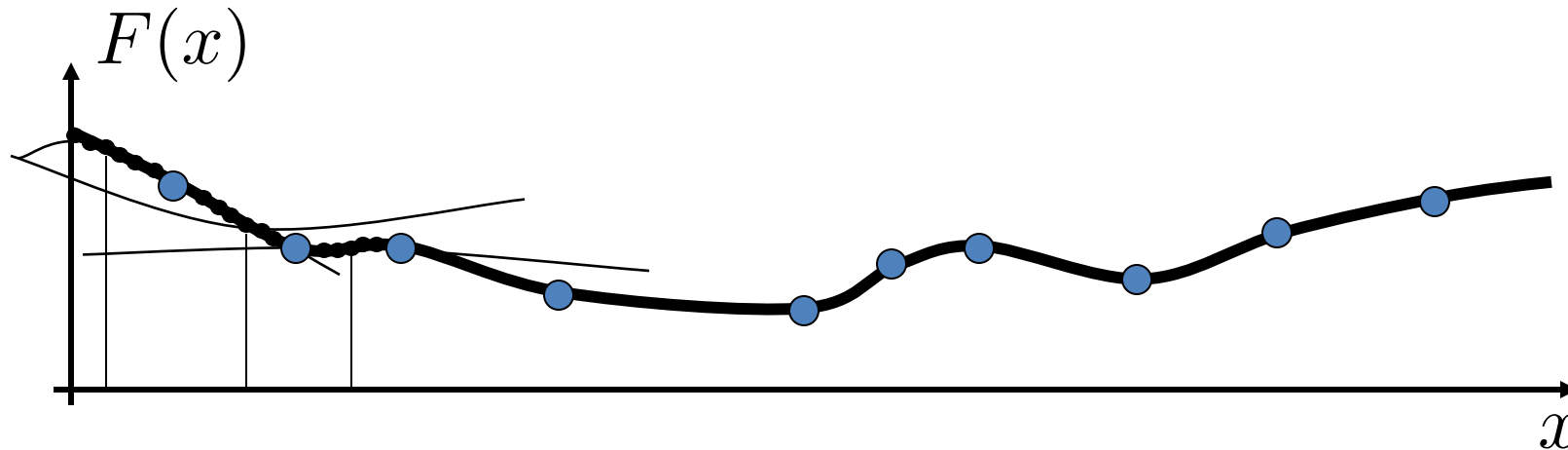
- Global approximation in  $\Pi_2^1$



$$f = \operatorname{argmin}_{f \in \Pi_2^1} \sum_{m=0}^{N-1} (f(c_m) - d_m)^2$$

# MLS - 1D Example

- MLS approximation using functions in  $\Pi_2^1$



$$F(x) = f_x(x), \quad f_x = \operatorname{argmin}_{f \in \Pi_2^1} \sum_{m=0}^{N-1} \theta(\|c_m - x\|) (f(c_m) - d_m)^2$$

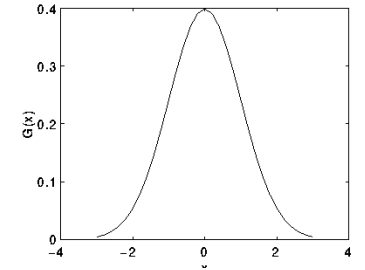


# Weight Functions

- Gaussian

- $h$  is a smoothing parameter

$$\theta(r) = e^{-\frac{r^2}{h^2}}$$



- Wendland function

- Defined in  $[0, h]$  and

$$\theta(r) = (1 - r/h)^4(4r/h + 1)$$

$$\theta(0) = 1, \theta(h) = 0, \theta'(h) = 0, \theta''(h) = 0$$

- “Singular” function

$$\theta(r) = \frac{1}{r^2 + \epsilon^2}$$

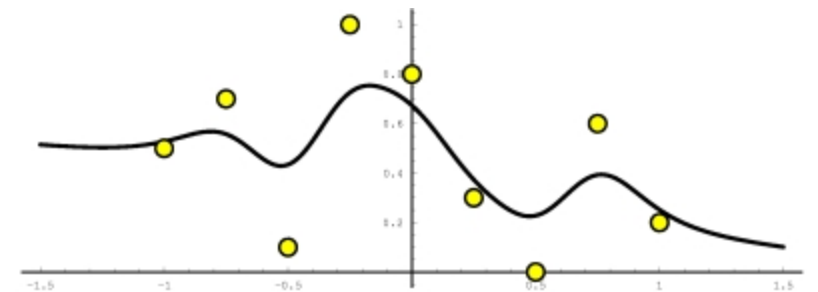
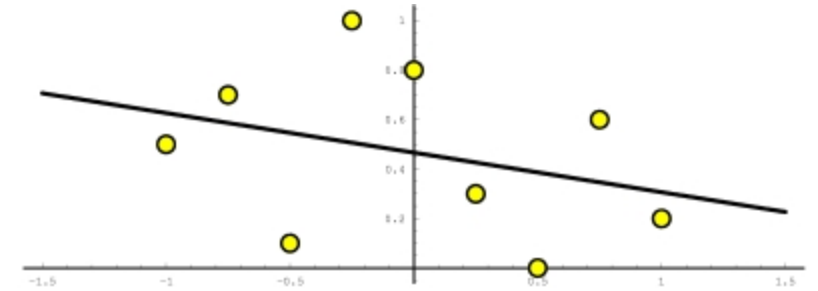
- For small  $\epsilon$ , weights are large near  $r=0$  (interpolation)

# Dependence on Weight Function

- Global least squares with linear polynomials
- MLS with (nearly) singular weight function
- MLS with approximating weight function

$$\theta(r) = \frac{1}{r^2 + \epsilon^2}$$

$$\theta(r) = e^{-\frac{r^2}{h^2}}$$

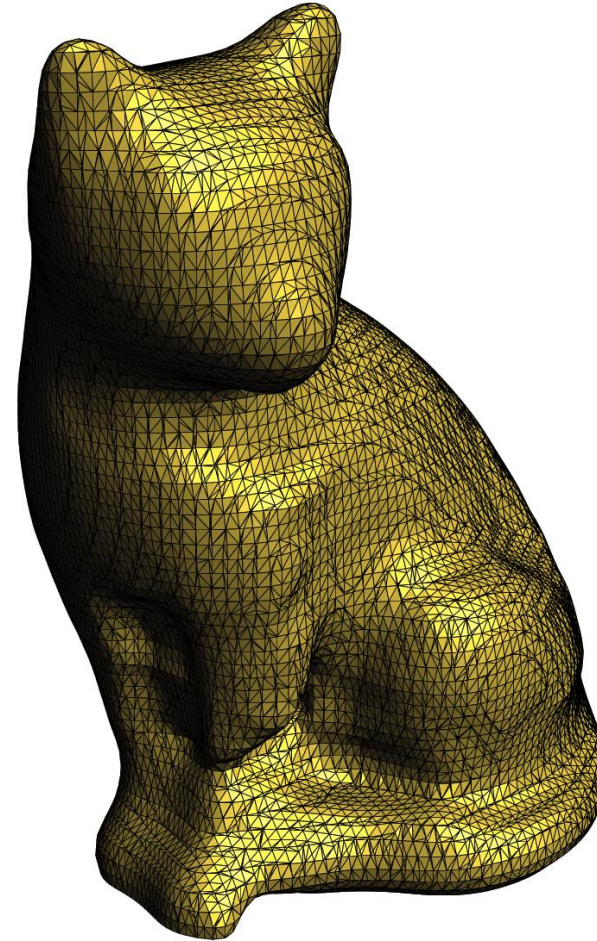
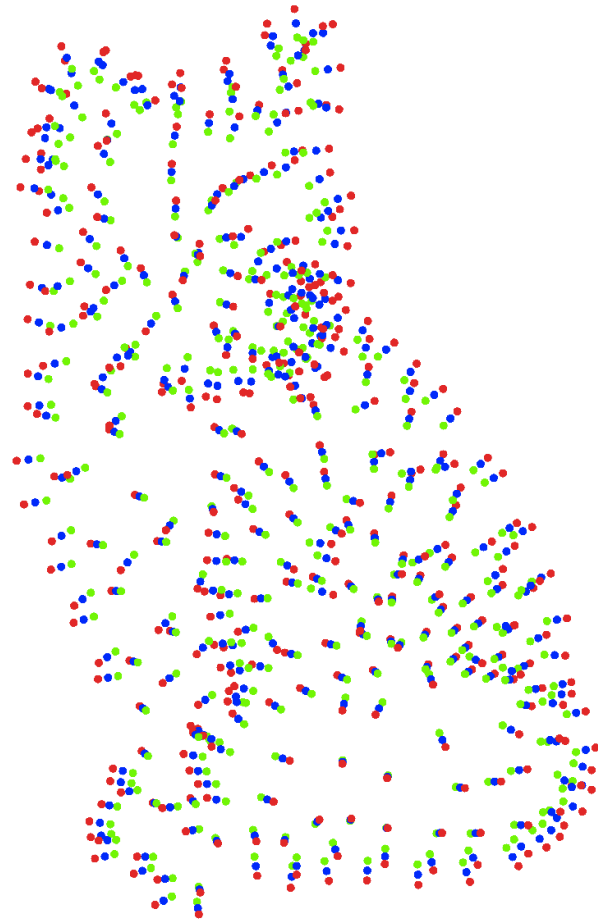


# Dependence on Weight Function

- The MLS function  $F$  is continuously differentiable if and only if the weight function  $\theta$  is continuously differentiable
- In general,  $F$  is as smooth as  $\theta$

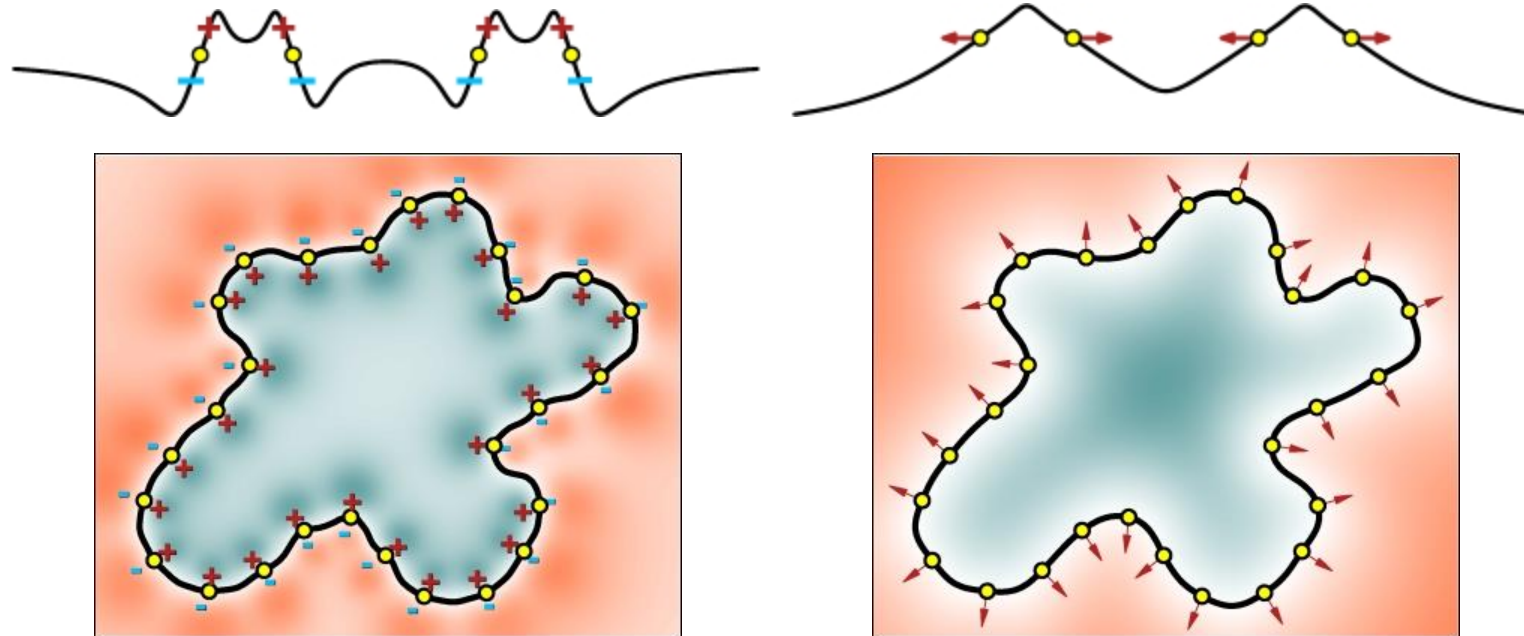
$$F(\mathbf{x}) = f_{\mathbf{x}}(\mathbf{x}), \quad f_{\mathbf{x}} = \operatorname{argmin}_{f \in \Pi_k^d} \sum_{m=0}^{N-1} \theta(\|\mathbf{c}_m - \mathbf{x}\|) (f(\mathbf{c}_m) - d_m)^2$$

# Example: Reconstruction



# MLS SDF - Possible Improvement

- Point constraints vs. true normal constraints



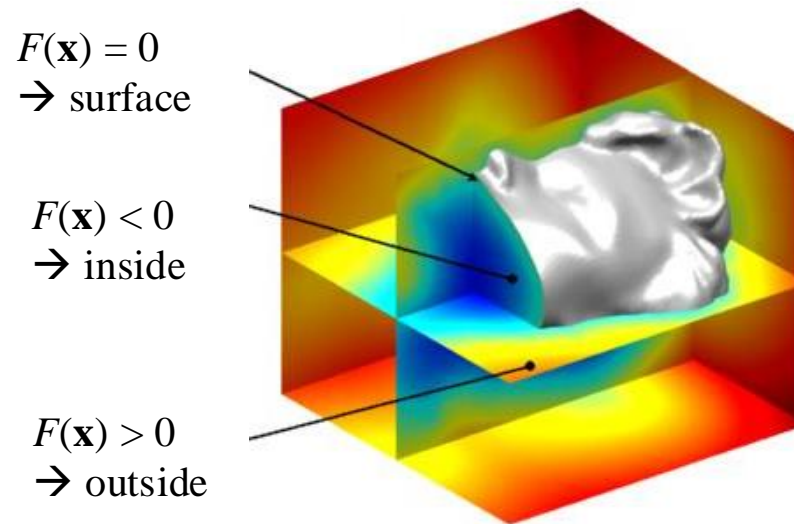
- Details: see [Shen et al. SIGGRAPH 2004] and Ex2

# Global RBF vs. Local MLS

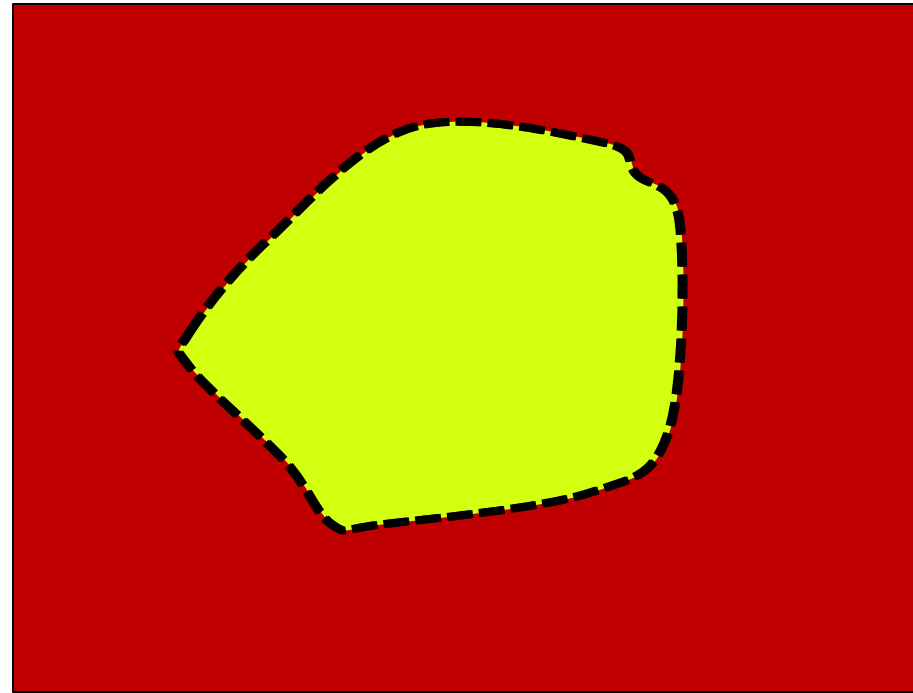
- RBF:
  - sees the whole data set, can make for very smooth surfaces
  - global (dense) system to solve - expensive
- MLS:
  - sees only a small part of the dataset, can get confused by noise
  - local linear solves - cheap

# Extracting the Surface

- Wish to compute a manifold mesh of the level set

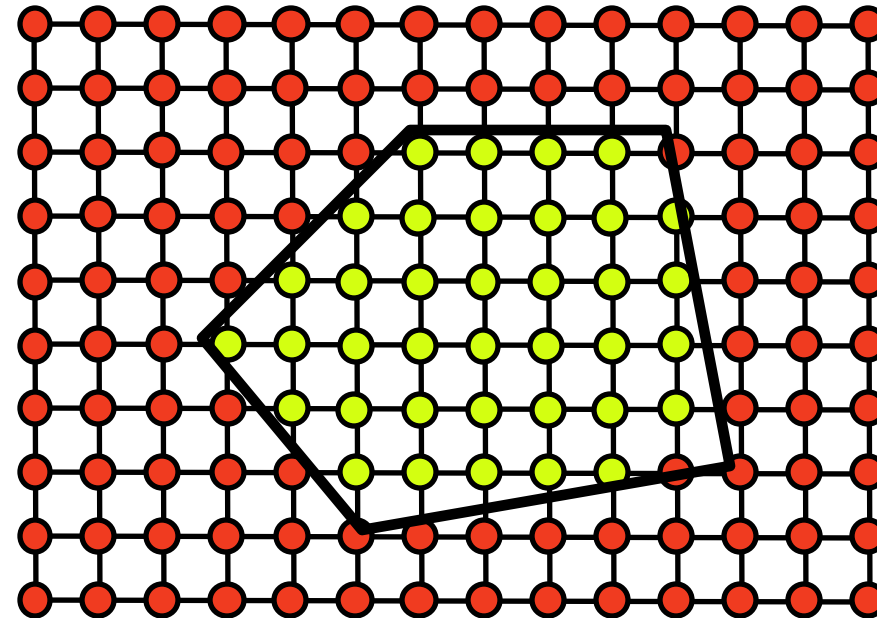


# Sample the SDF



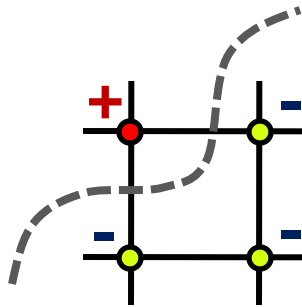


# Sample the SDF

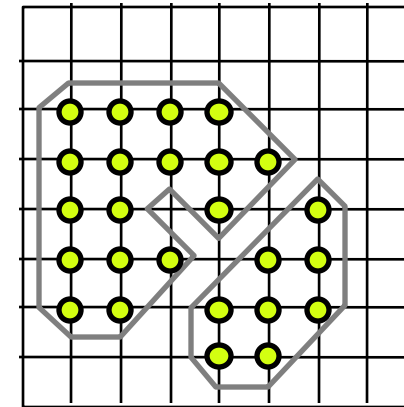
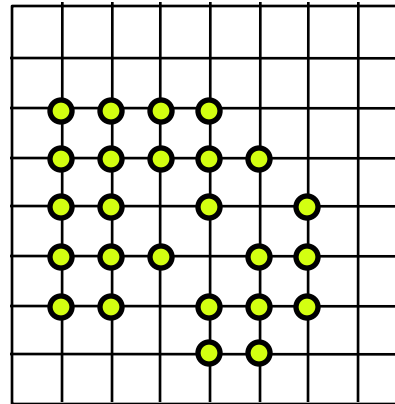


# Tessellation

- Want to approximate an implicit surface with a mesh
- Can't explicitly compute all the roots
  - Sampling the level set is difficult (root finding)
- Solution: find approximate roots by trapping the implicit surface in a grid (lattice)

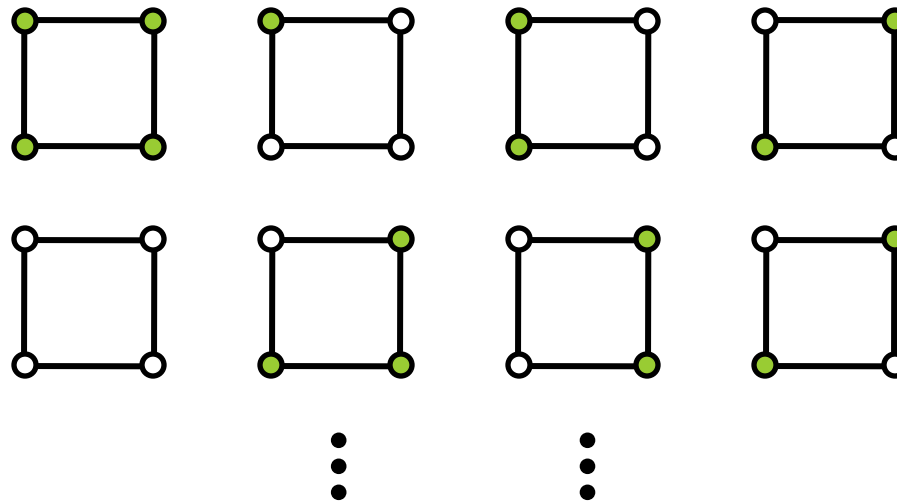


$$\bullet F(\mathbf{x}) < 0$$



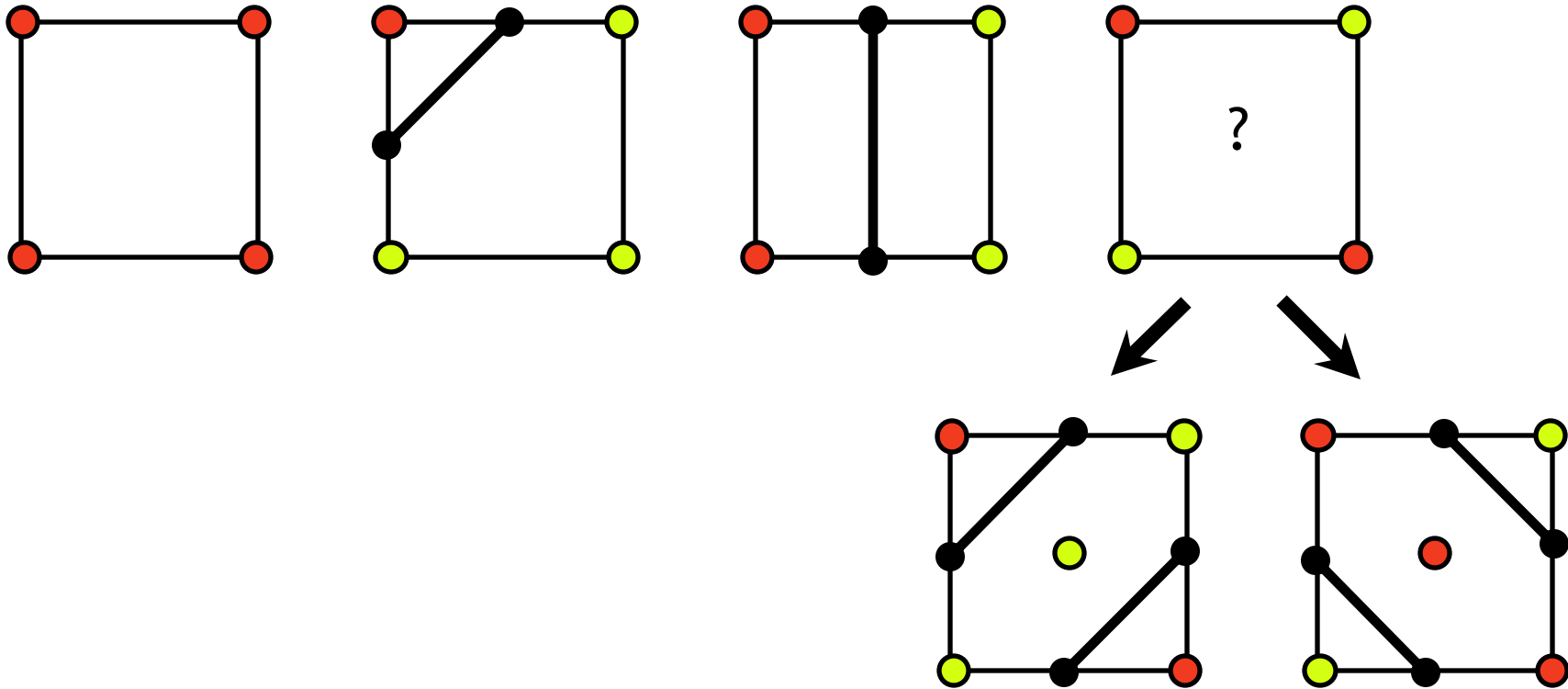
# Marching Squares

- 16 different configurations in 2D
- 4 equivalence classes (up to rotational and reflection symmetry + complement)



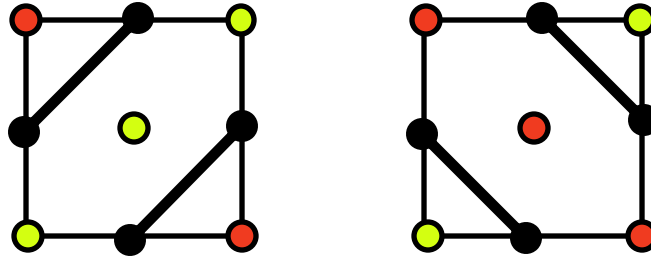
# Tessellation in 2D

- 4 equivalence classes (up to rotational and reflection symmetry + complement)

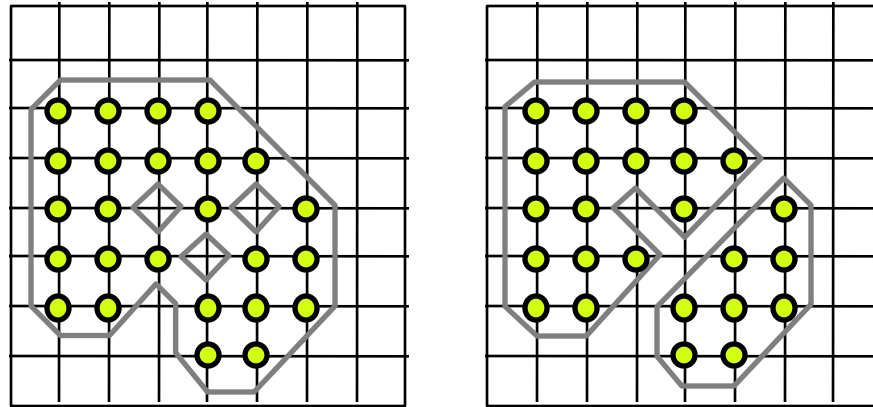


# Tessellation in 2D

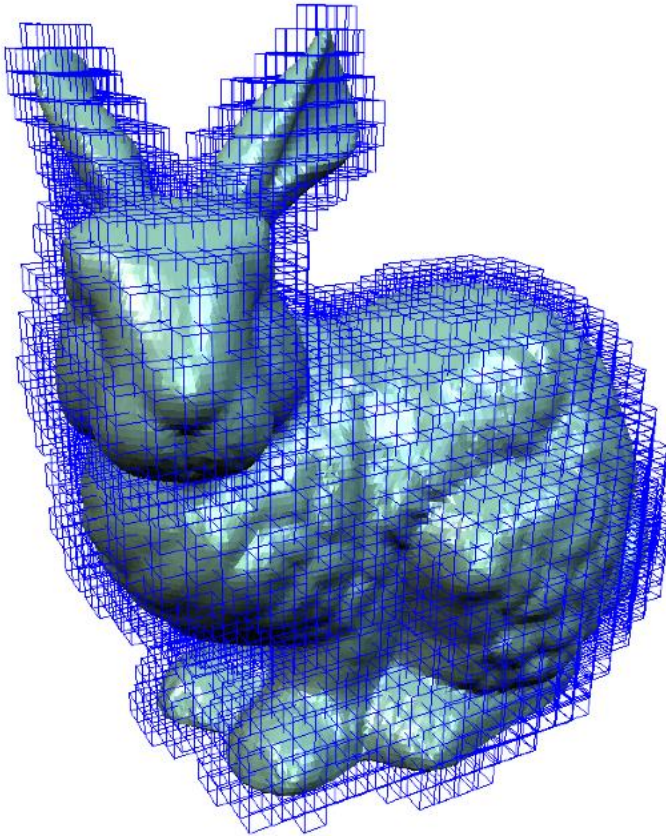
- Case 4 is ambiguous:



- Always pick consistently to avoid problems with the resulting mesh

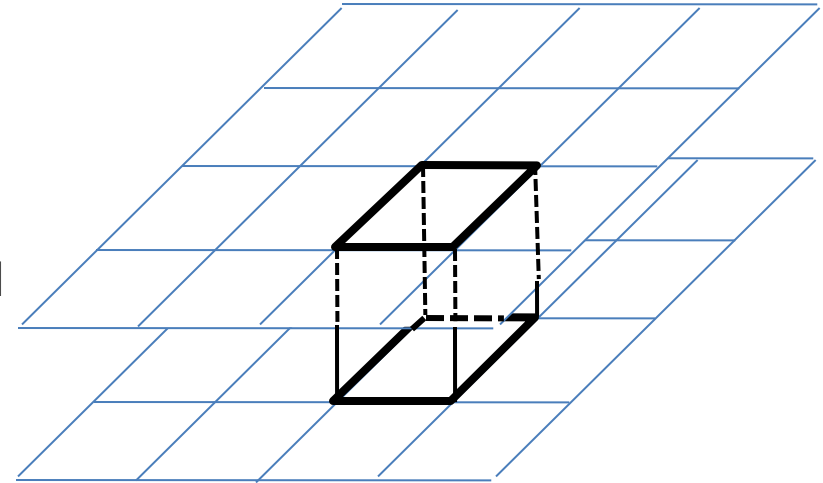


# 3D: Marching Cubes



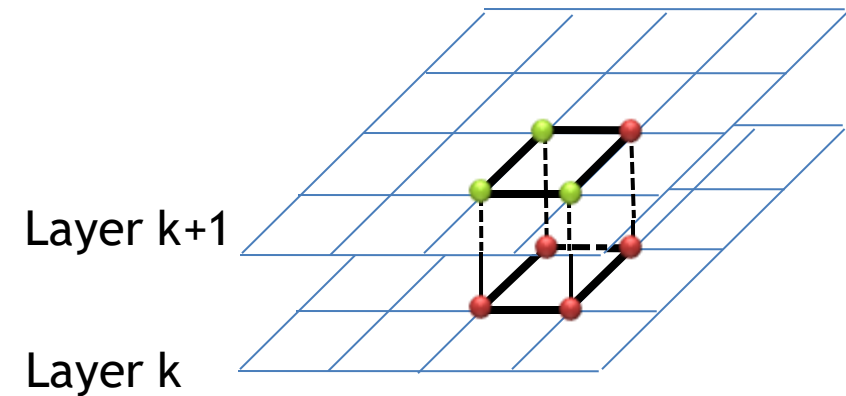
Layer  $k+1$

Layer  $k$



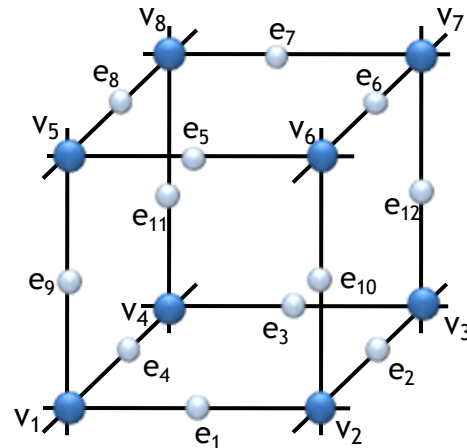
# Marching Cubes

- Marching Cubes (Lorensen and Cline 1987)
  1. Load 4 layers of the grid into memory
  2. Create a cube whose vertices lie on the two middle layers
  3. Classify the vertices of the cube according to the implicit function (inside, outside or on the surface)



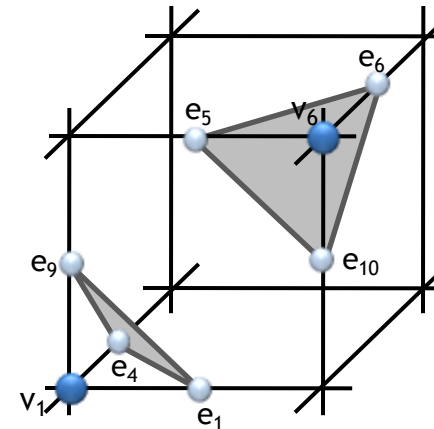
# Marching Cubes

4. Compute case index. We have  $2^8 = 256$  cases (0/1 for each of the eight vertices) - can store as 8 bit (1 byte) index.



index = 

$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$	$v_7$	$v_8$
-------	-------	-------	-------	-------	-------	-------	-------



index = 

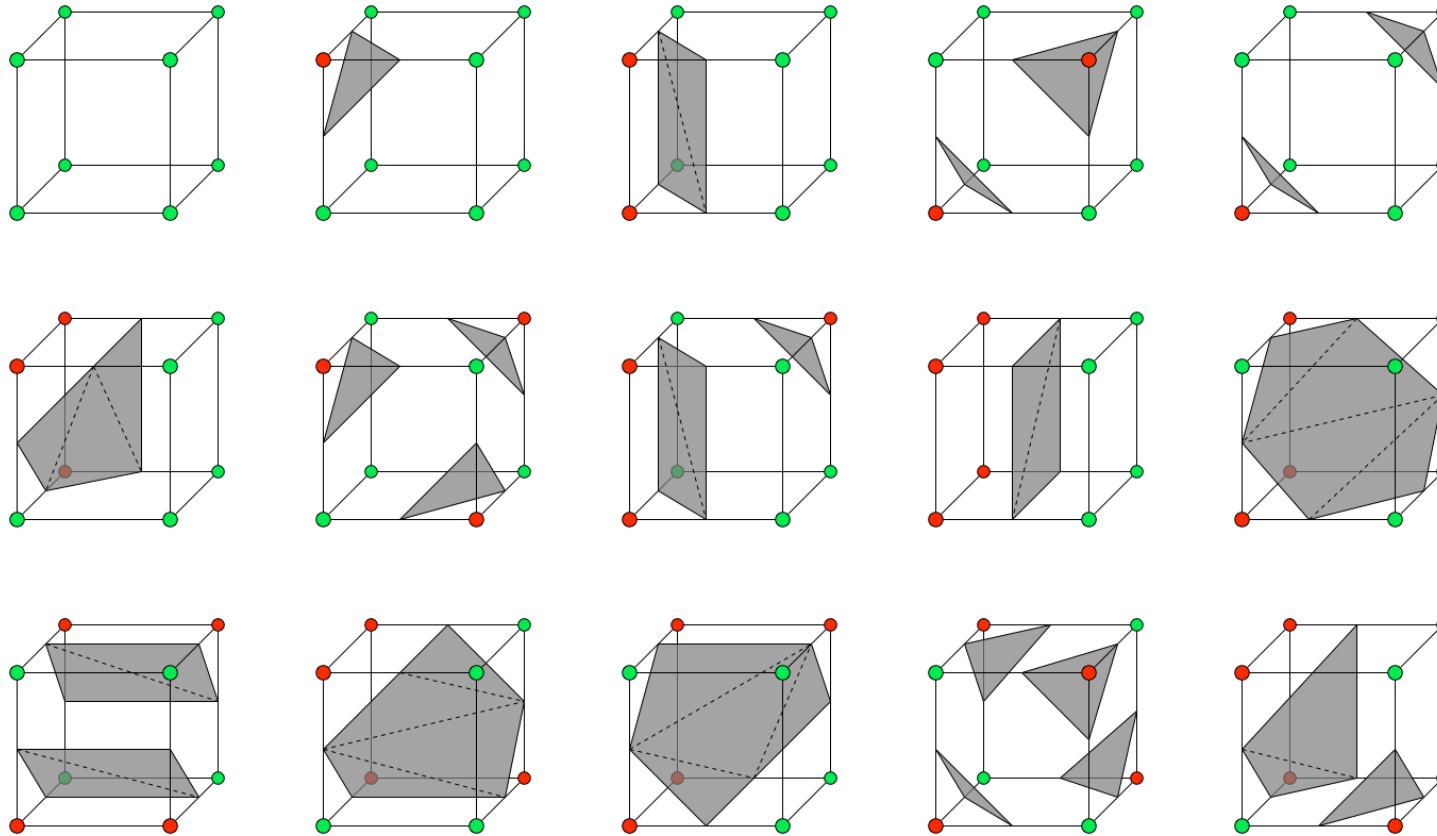
0	0	1	0	0	0	0	1
---	---	---	---	---	---	---	---

 = 33



# Marching Cubes

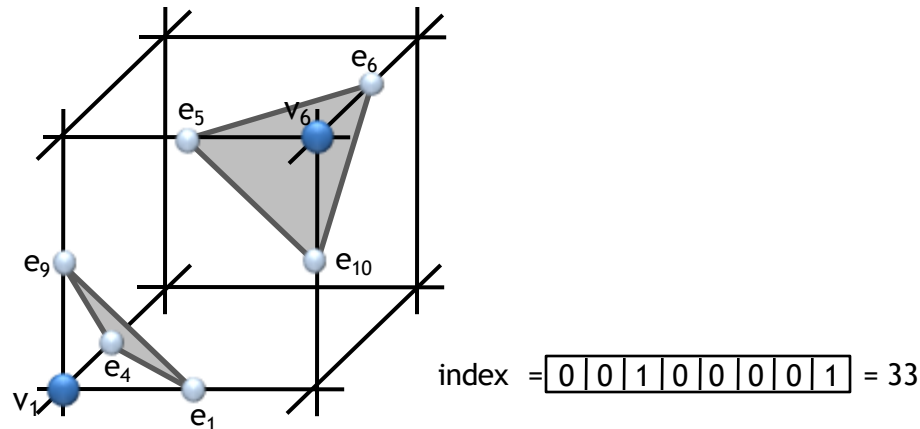
- Unique cases (by rotation, reflection and complement)



# Tessellation

## 3D - Marching Cubes

5. Using the case index, retrieve the connectivity in the look-up table
- Example: the entry for index 33 in the look-up table indicates that
  - the cut edges are  $e_1$ ;  $e_4$ ;  $e_5$ ;  $e_6$ ;  $e_9$  and  $e_{10}$  ;
  - the output triangles are  $(e_1; e_9; e_4)$  and  $(e_5; e_{10}; e_6)$ .



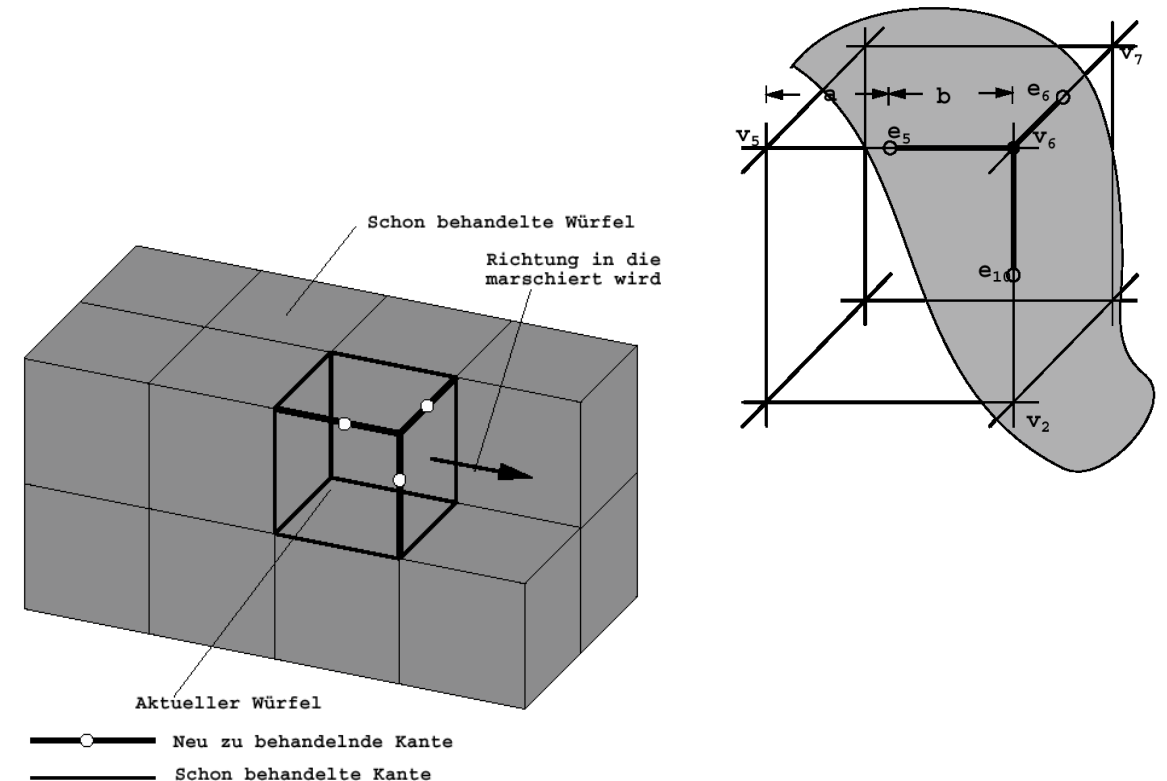
# Marching Cubes

6. Compute the position of the cut vertices by linear interpolation:

$$\mathbf{v}_s = t\mathbf{v}_a + (1 - t)\mathbf{v}_b$$

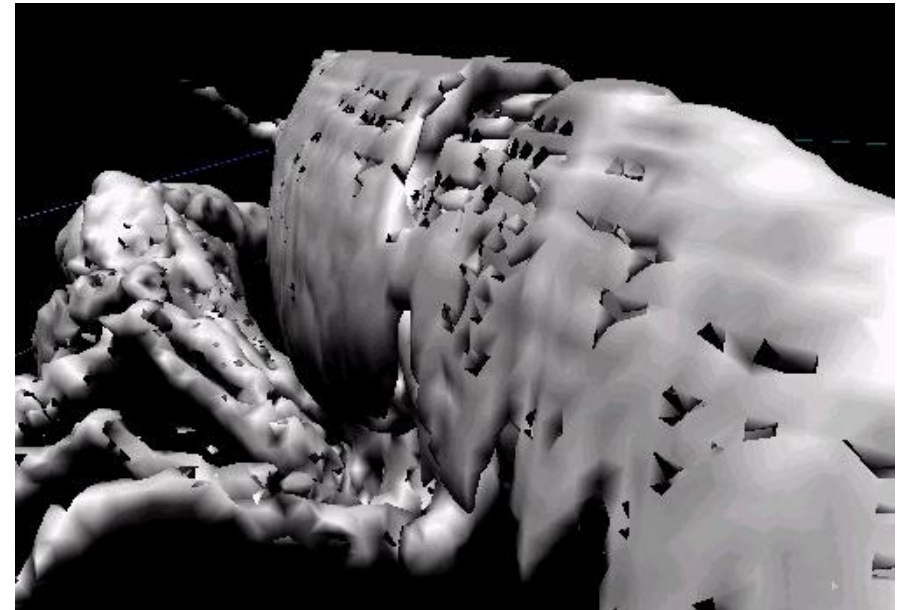
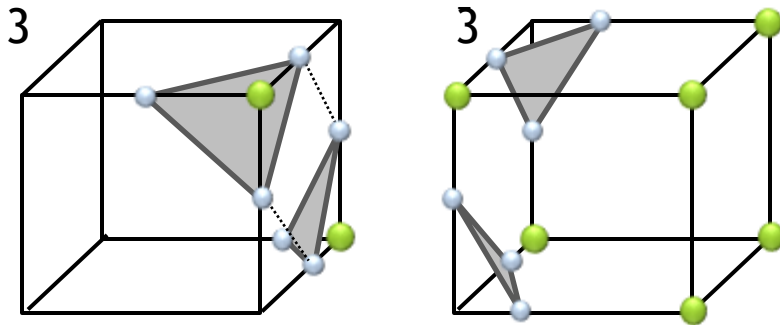
$$t = \frac{F(\mathbf{v}_b)}{F(\mathbf{v}_b) - F(\mathbf{v}_a)}$$

7. Move to the next cube



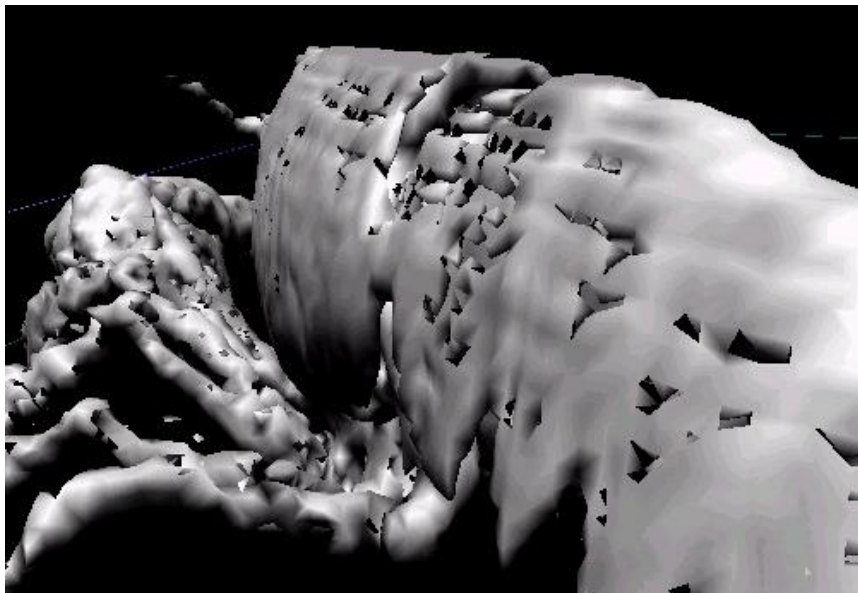
# Marching Cubes - Problems

- Have to make consistent choices for neighboring cubes - otherwise get holes

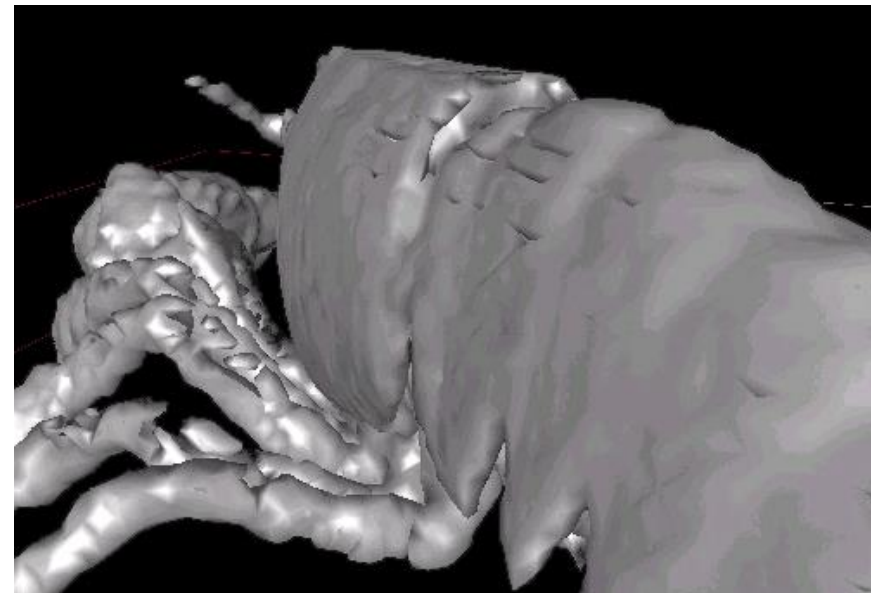


# Marching Cubes - Problems

- Resolving ambiguities



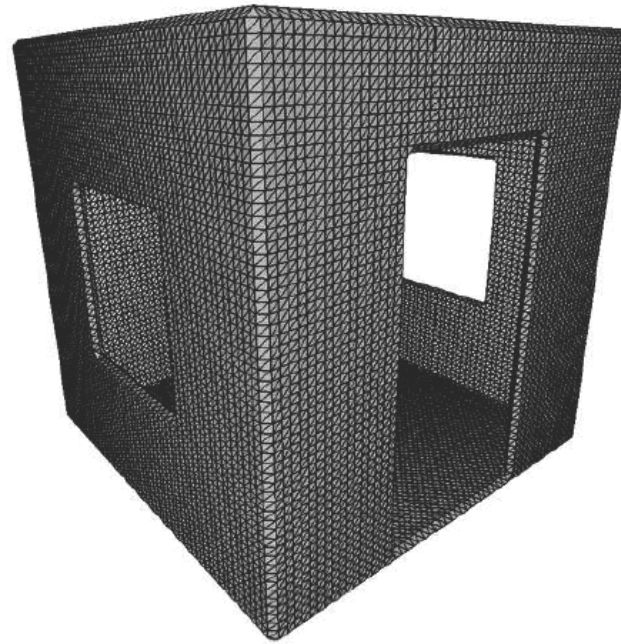
Ambiguity



No Ambiguity

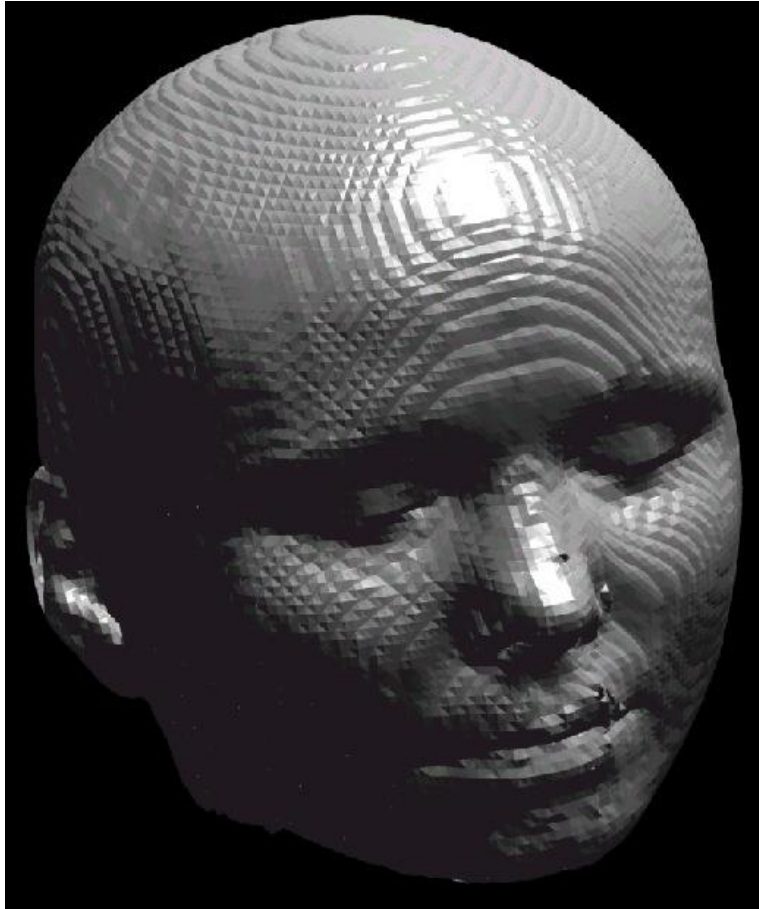
# Marching Cubes - Problems

- Grid not adaptive
- Many polygons required to represent small features



Images from: "Dual Marching Cubes: Primal Contouring of Dual Grids"  
by Schaeffer et al.

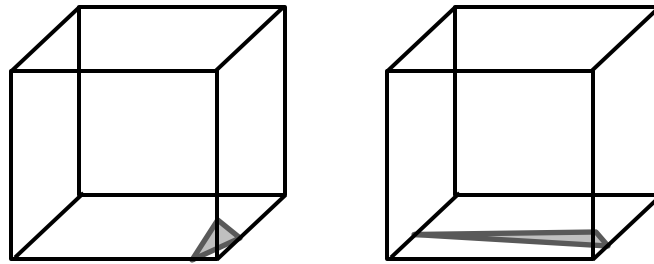
# Marching Cubes - Problems





# Marching Cubes - Problems

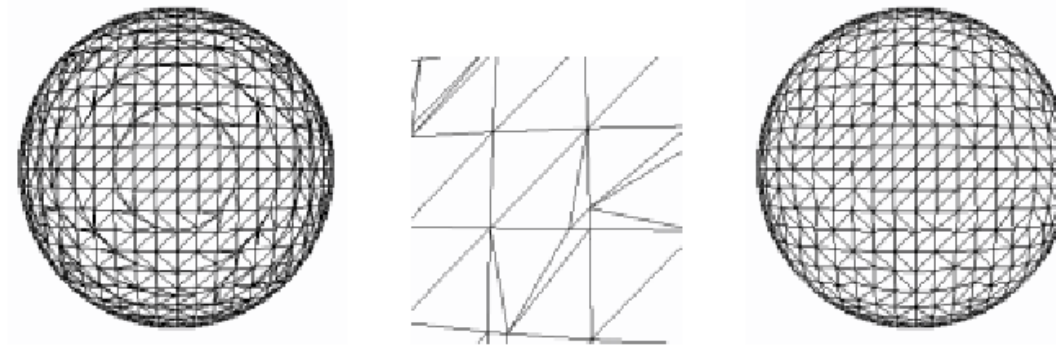
- Problems with short triangle edges
  - When the surface intersects the cube close to a corner, the resulting tiny triangle doesn't contribute much area to the mesh
  - When the intersection is close to an edge of the cube, we get skinny triangles (bad aspect ratio)
- Triangles with short edges waste resources but don't contribute to the surface mesh representation





# Grid Snapping

- Solution: threshold the distances between the created vertices and the cube corners
- When the distance is smaller than  $d_{\text{snap}}$  we snap the vertex to the cube corner
- If more than one vertex of a triangle is snapped to the same point, we discard that triangle altogether



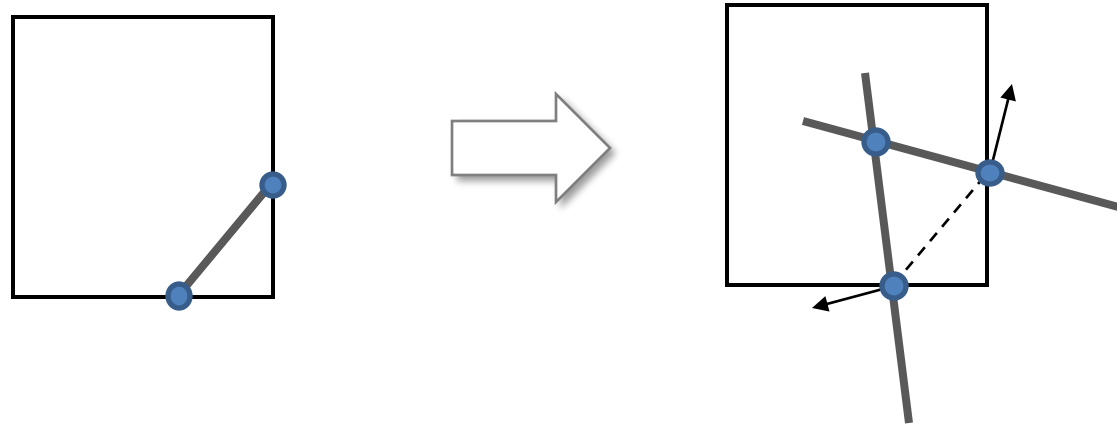
# Grid Snapping

- With grid snapping one can obtain significant reduction of space consumption

$d_{\text{snap}}$	0	0,1	0,2	0,3	0,4	0,46	0,495
Vertices	1446	1398	1254	1182	1074	830	830
Reduction (%)	0	3,3	13,3	18,3	25,7	42,6	42,6

# Sharp Corners and Features

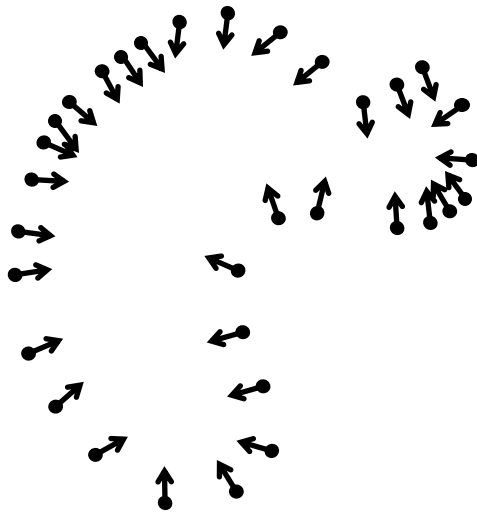
- Kobbelt et al. SIGGRAPH 2001 „Feature sensitive surface extraction from volume data“
  - Evaluate the normals (use gradient of  $F$ )
  - When they significantly differ, create an additional vertex



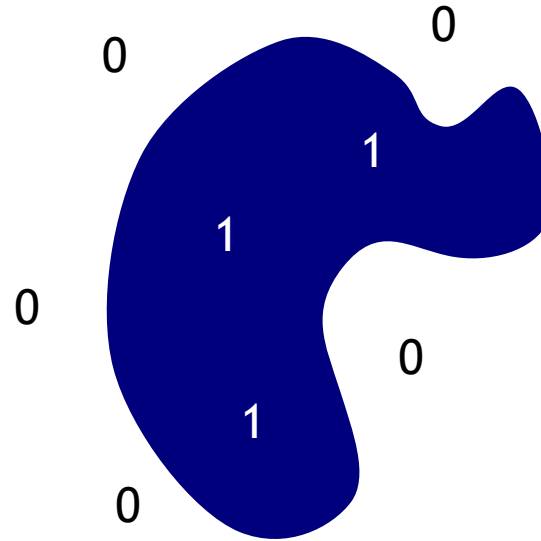
# Poisson Surface Reconstruction

- Very popular modern method, code available:  
M. Kazhdan, M. Bolitho and H. Hoppe, Symposium on Geometry Processing 2006, follow-up in 2013, 2020...  
<http://www.cs.jhu.edu/~misha/Code/PoissonRecon/>
- Global fitting of an *indicator function* using a PDE
  - Robust to noise, sparse, computationally tractable
- You will try out the code in Ex2 and compare with MLS results

# Poisson Surface Reconstruction



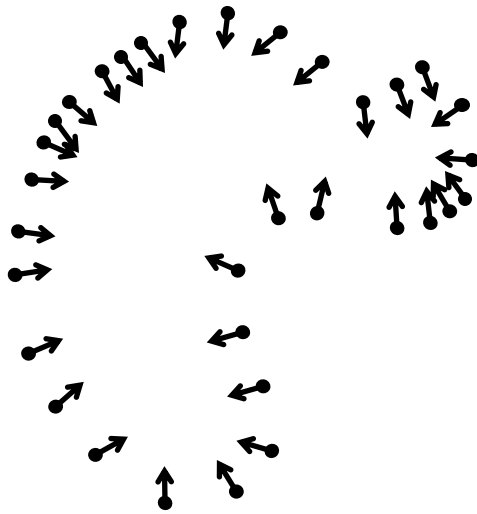
Oriented points



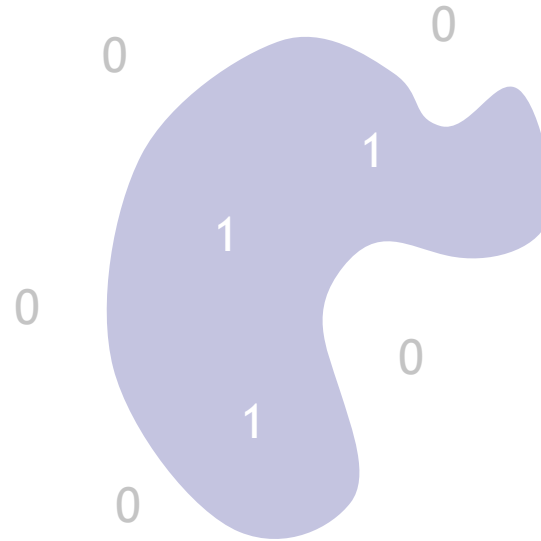
Indicator function

$$\chi_M$$

# Poisson Surface Reconstruction



Oriented points

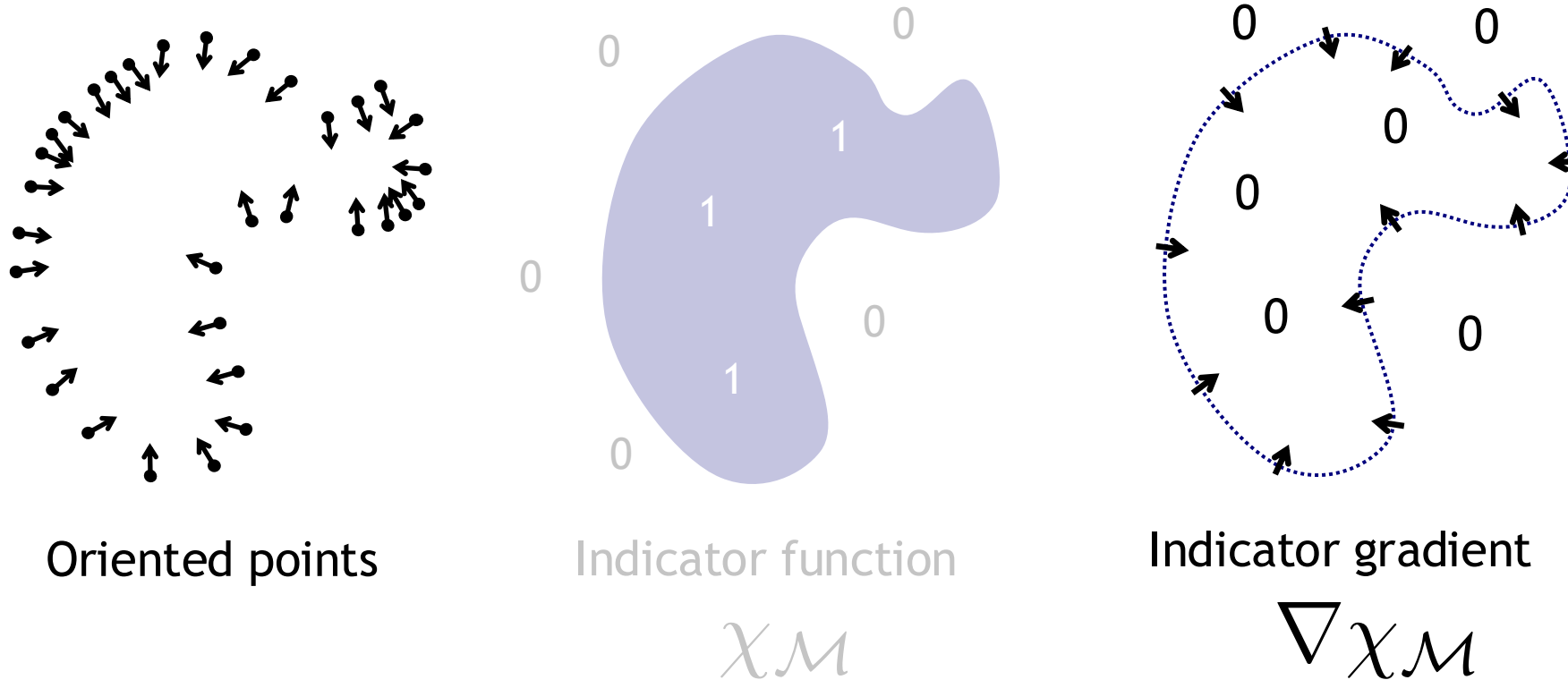


Indicator function

$\chi_M$

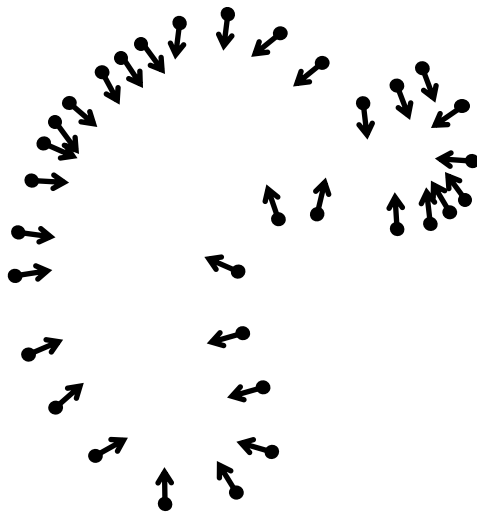
We don't know the indicator function ☹

# Poisson Surface Reconstruction

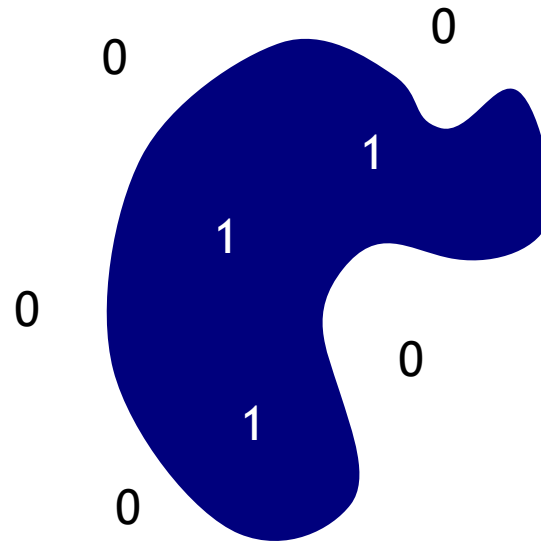


But we can estimate its gradient! ☺

# Poisson Surface Reconstruction

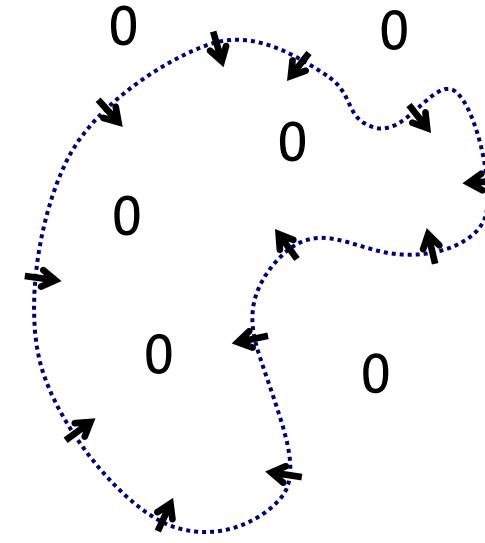


Oriented points



Indicator function

$$\chi_{\mathcal{M}}$$



Indicator gradient

$$\nabla \chi_{\mathcal{M}}$$

Reconstruct  $\chi$  by solving the Poisson equation

$$\Delta \chi_{\mathcal{M}} = \operatorname{div} \nabla \chi_{\mathcal{M}}$$

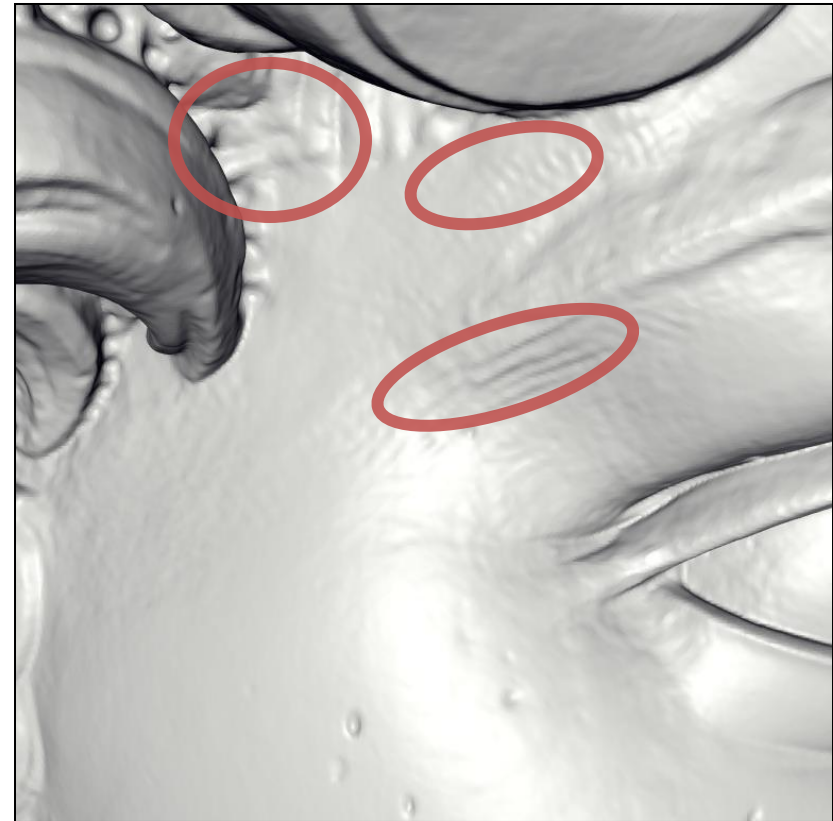


# Michelangelo's David

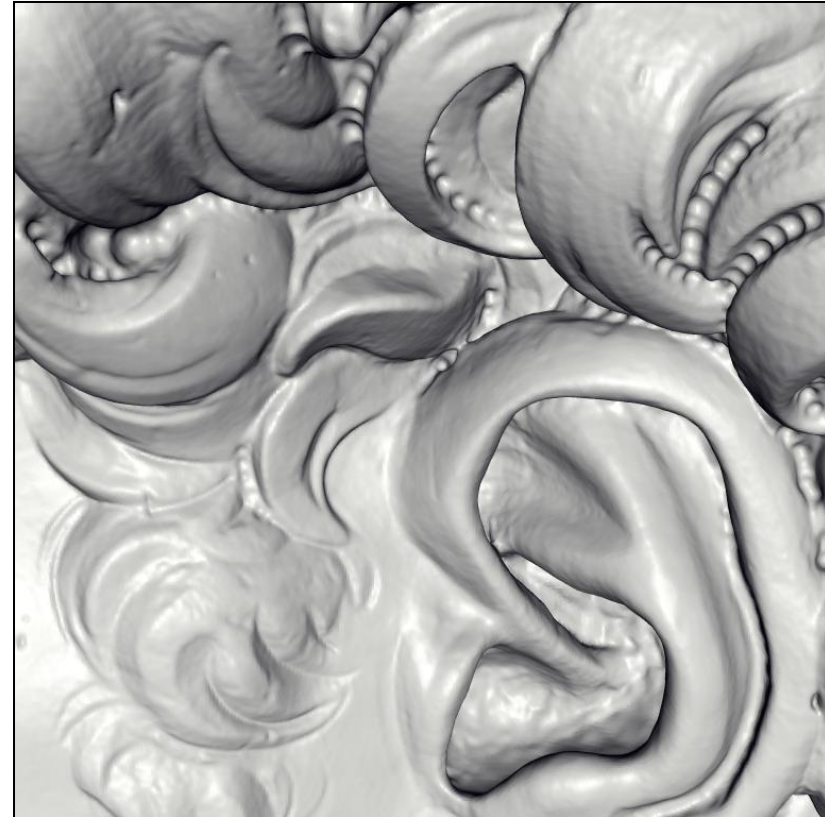


- 215M data points from 1000 scans
- 22M triangle reconstruction
- Compute time: 2.1 hours (this was in year 2006)
- Peak memory: 6600 MB

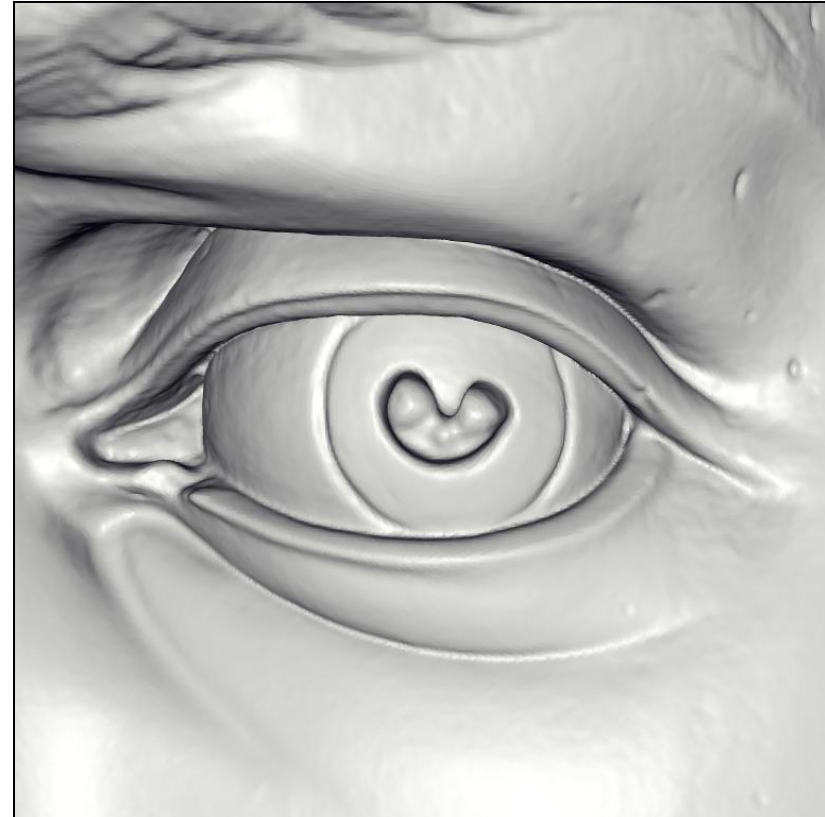
# David - Chisel marks



# David - Drill Marks



# David - Eye



# Thank You

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