252-0538-00L, Spring 2025

Shape Modeling and Geometry Processing

Mesh Editing I: Introduction; Differential Deformations

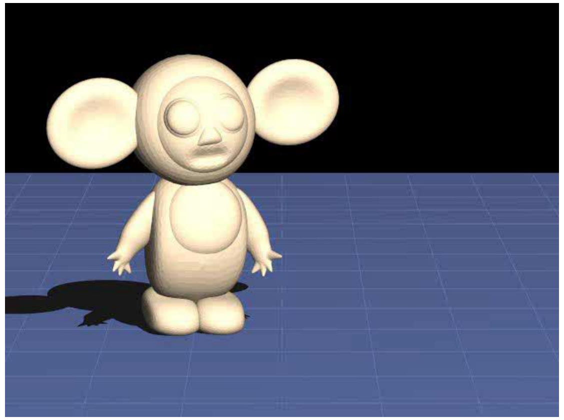
INTERACTIVE GEOMETRY LAB

April 30, 2025

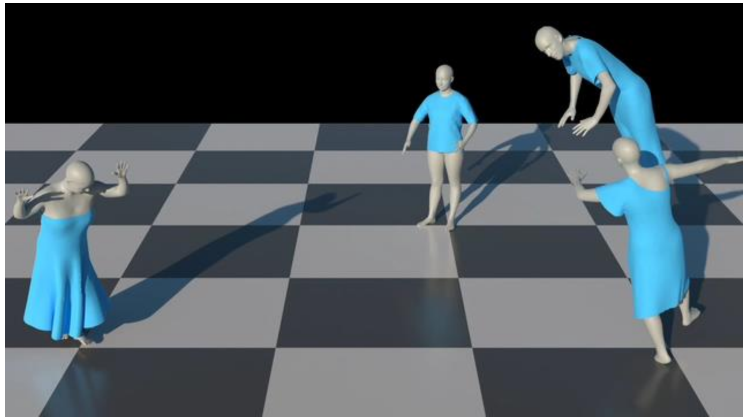
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Why Shape Deformation?

- Animation

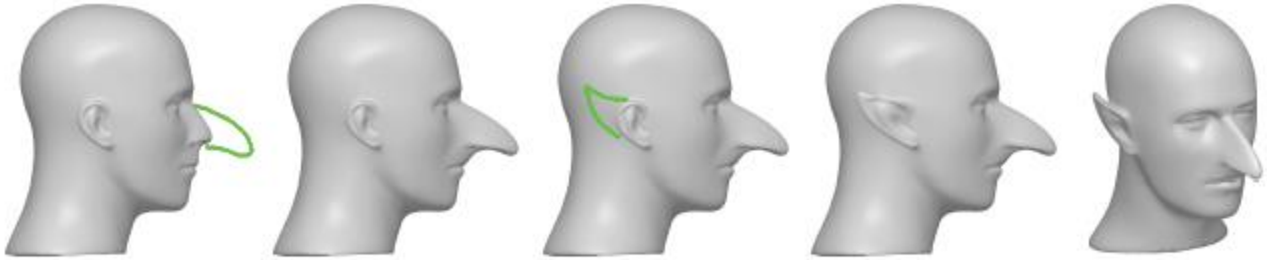


Baran and Popovic, SIGGRAPH 2007



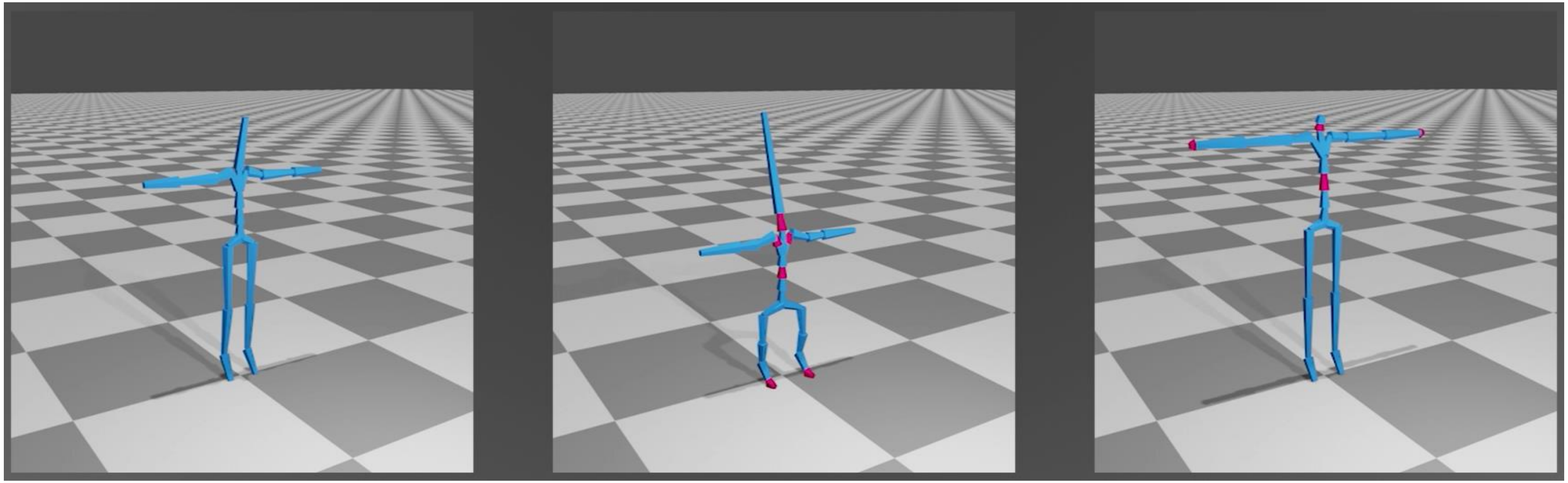
Li et al. EG 2024

- Editing



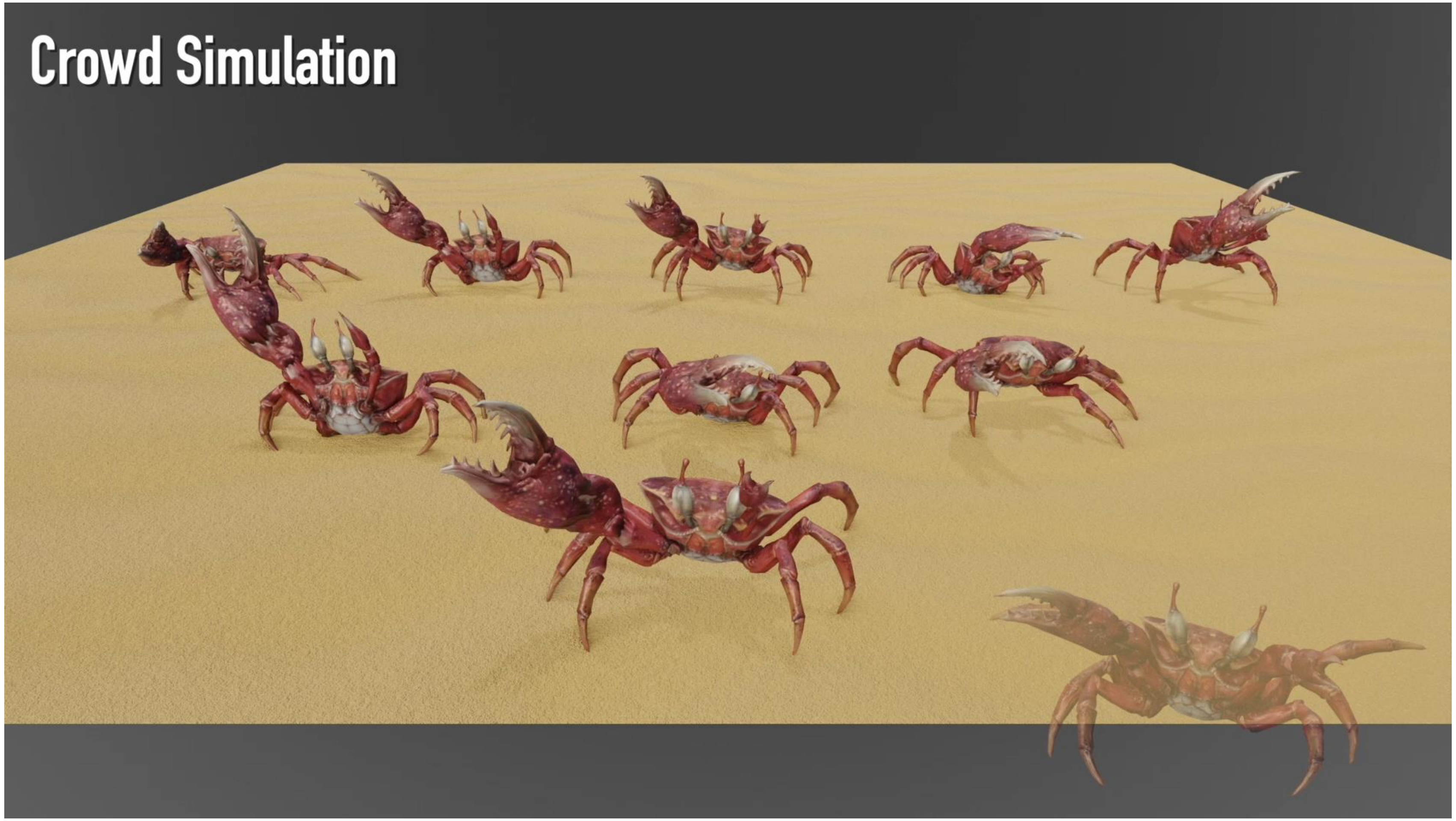
Zimmermann et al., SBIM 2007

Animation: motion and skin



Aberman, Li, et al. SIGGRAPH 2020

Animation: motion and skin

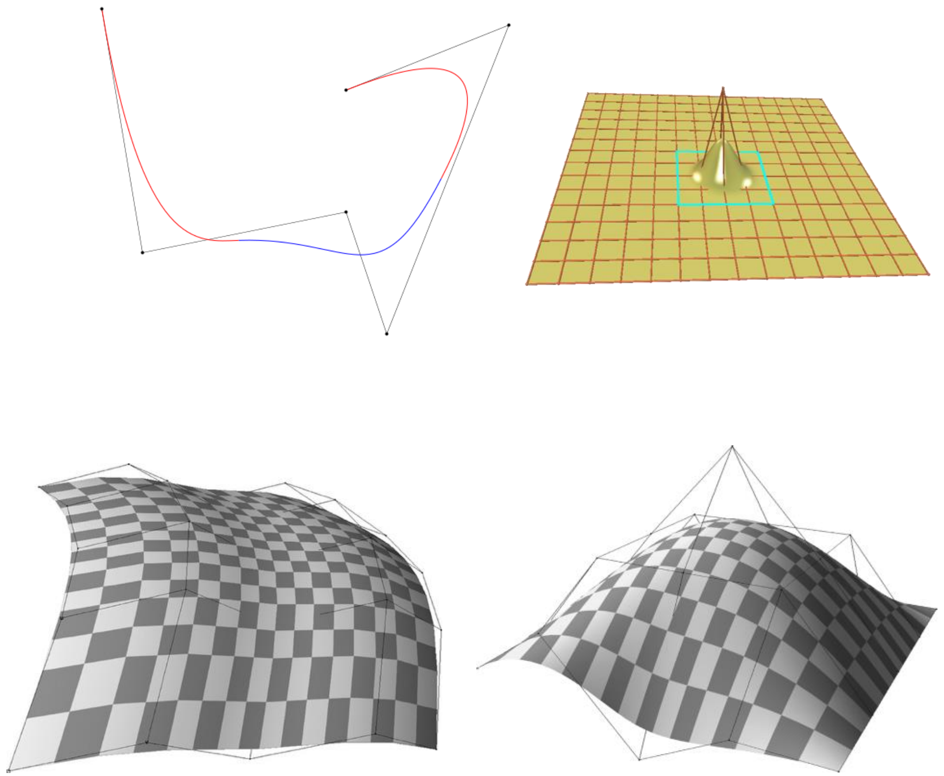


Li et al. SIGGRAPH 2022

Parametric Curves and Surfaces

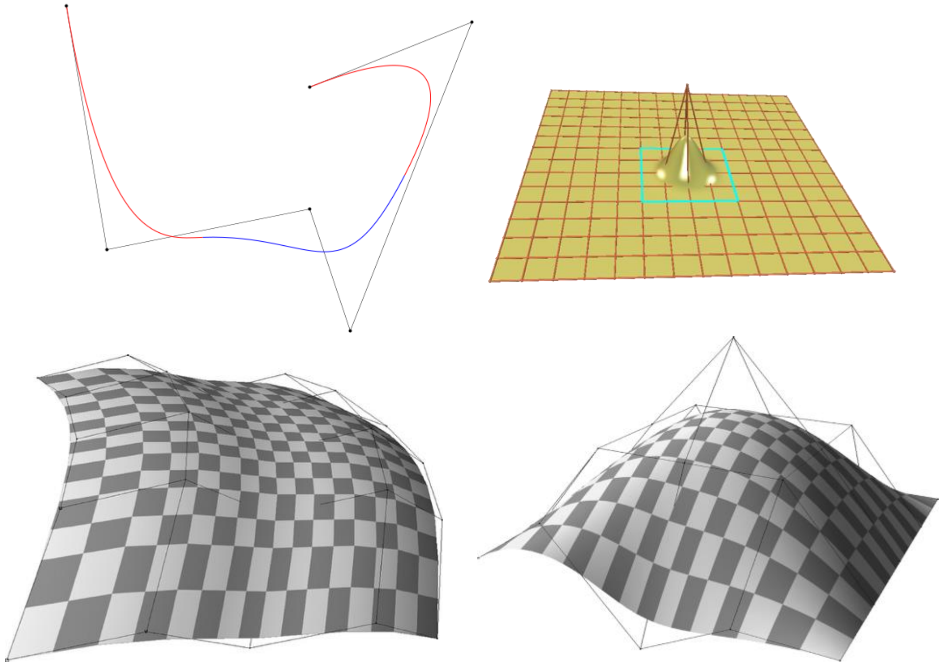
- Deformation by control point manipulation
- Built-in deformation mechanism

- Control structure is pre-set (can’t pull on arbitrary points)



Traditional CAD vs Meshes

$$\mathbf{x}(u,v) = \sum_{i,j} \mathbf{p}_{i,j} B_i(u) B_j(v)$$



$$\min_{\mathbf{x}} \ E(\mathbf{x}) \quad s.t. \ \mathbf{x}|_{\mathcal{C}} = \mathbf{x}_{\text{fixed}}$$

User has more freedom!
Select and manipulate arbitrary regions.

Traditional CAD vs Meshes

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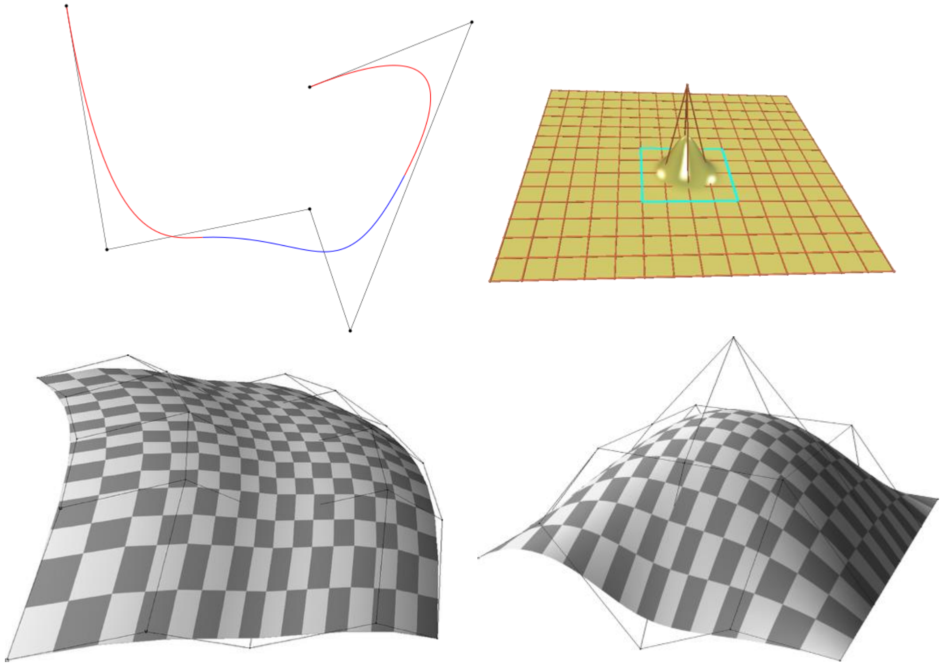


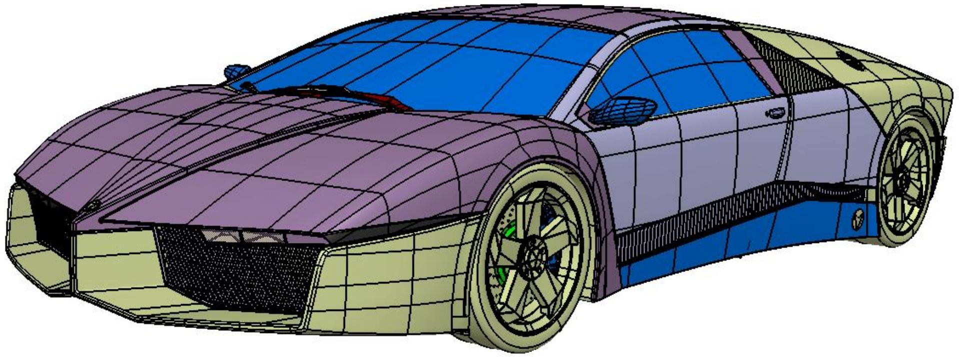
$$\min_{\mathbf{x}} \; E(\mathbf{x}) \quad s.t. \; \mathbf{x}|_{\mathcal{C}} = \mathbf{x}_{\text{fixed}}$$

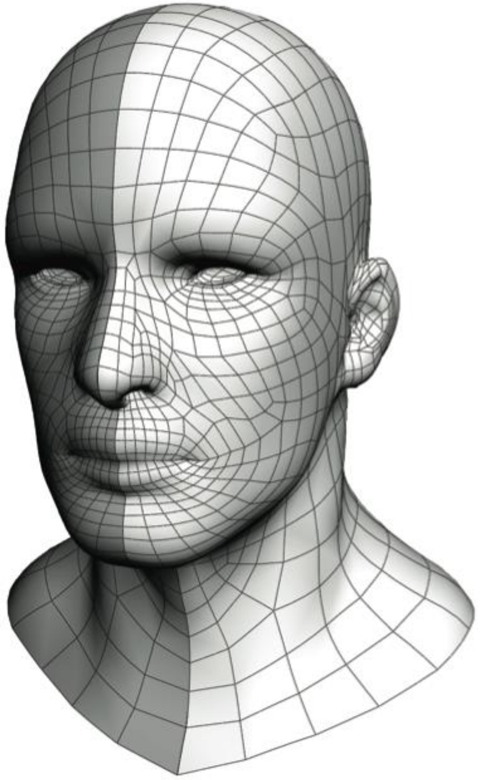
User has more freedom!
Select and manipulate arbitrary regions.

Parametric Curves and Surfaces

- Hard to change / adapt control structure to user needs

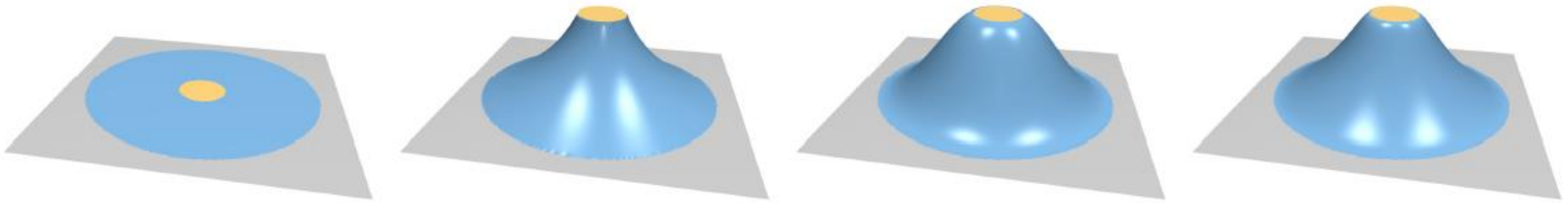
- Hard to experiment, need a precise idea of what will be modeled



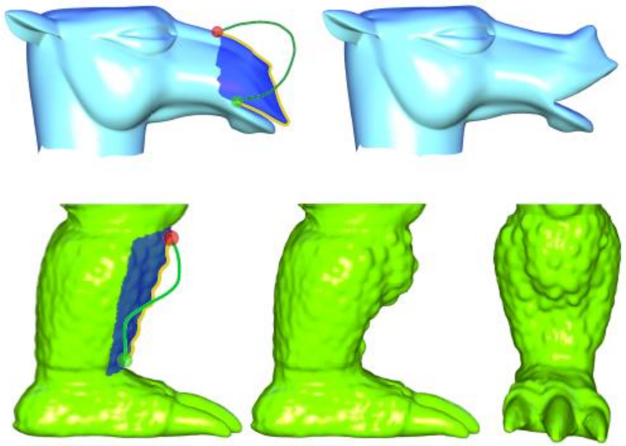


Surface-based Deformations: Examples

- Region of interest (ROI) + affine deformation handle with variable boundary continuity

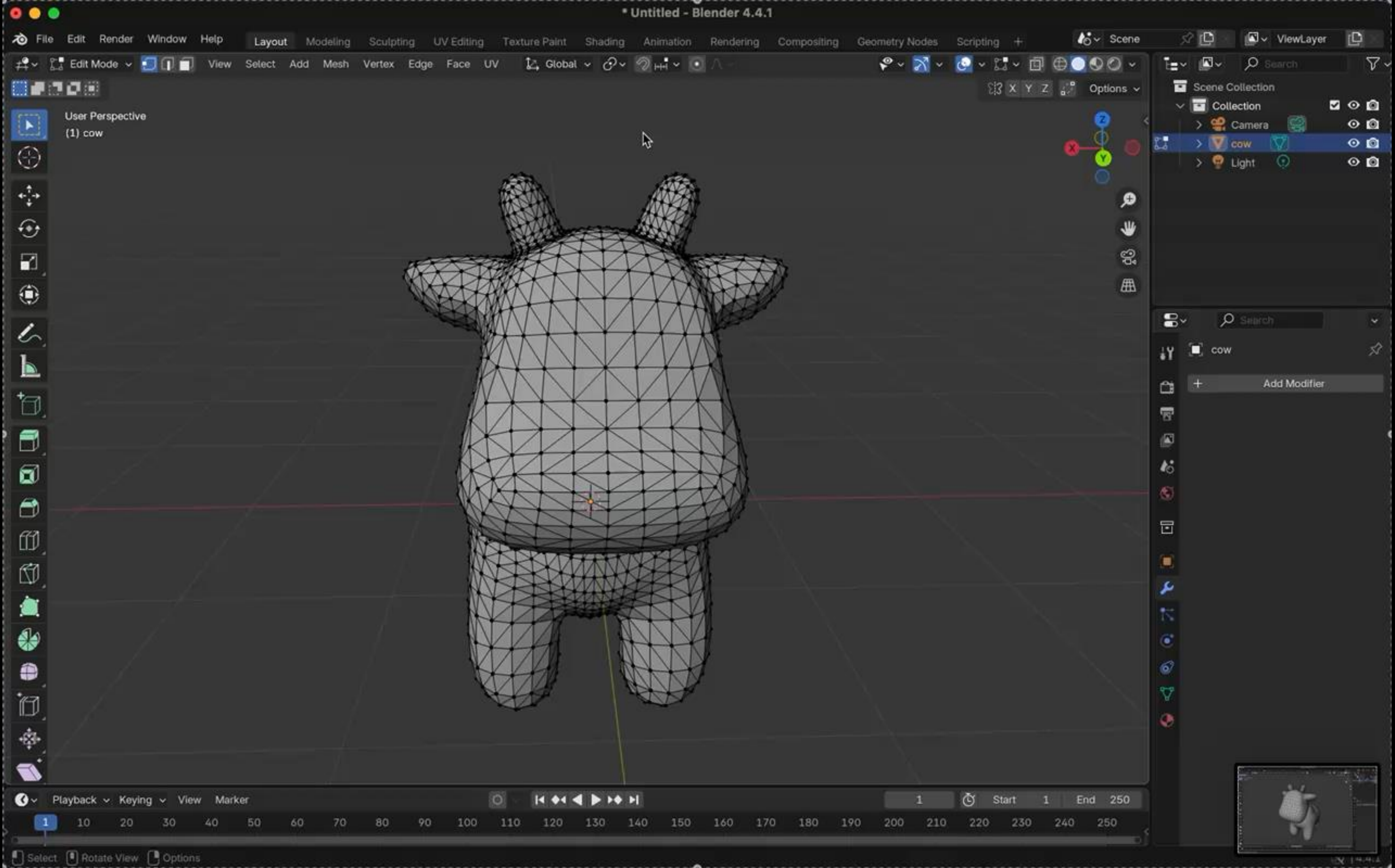


- Intuitive sketch-based deformation interfaces



Mesh Deformation

- Naïve method: dragging single vertices



* Untitled - Blender 4.4.1
File
Edit
Render
Window
Help
Layout
Modeling
Sculpting
UV Editing
Texture Paint
Shading
Animation
Rendering
Compositing
Geometry Nodes
Scripting
Scene
ViewLayer
Edit Mode
View
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Scene Collection
Collection
Camera
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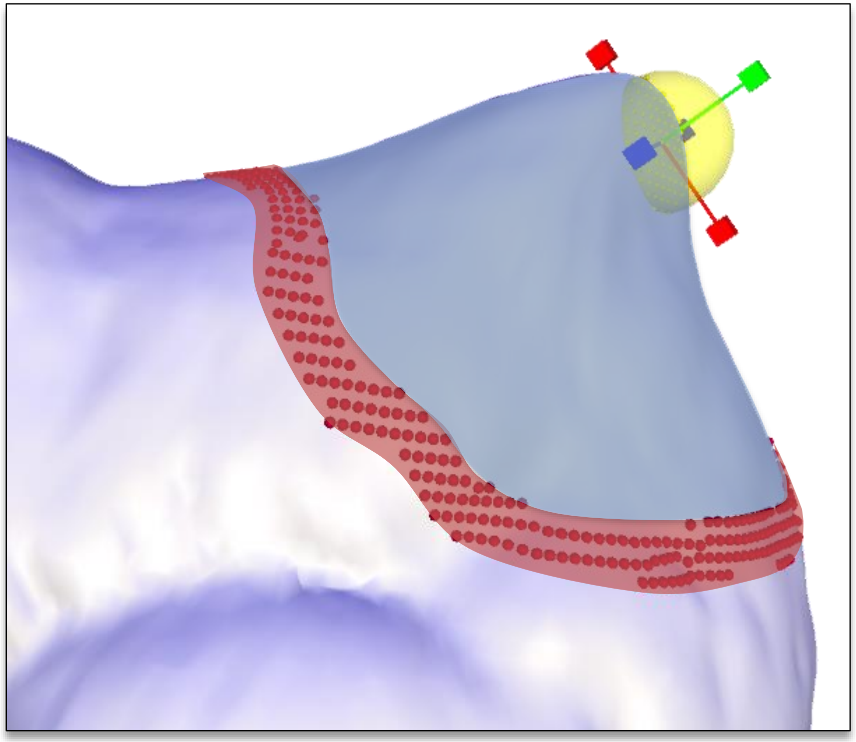
Light
cow
Add Modifier
Playback
Keying
View
Marker
Start 1
End 250
Select
Rotate View
Options

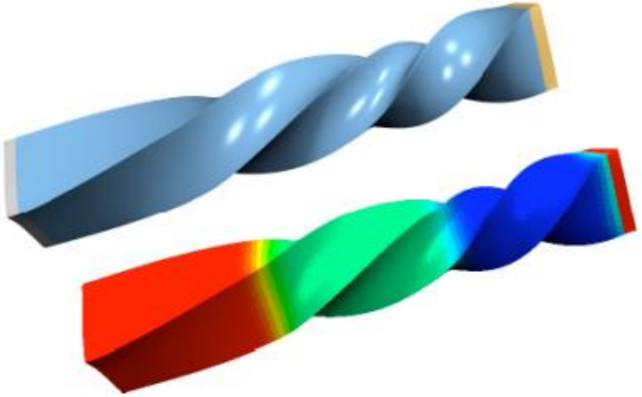
Mesh Deformation

- Naïve method: dragging single vertices
- Smarter:
  - Introduce a small set of deformation handles
    - Makes deformation/editing easier
    - Introduces a trade-off between degrees of freedom and simplicity of the deformation task
  - Create a small set of control parameters
    - Affine transformations

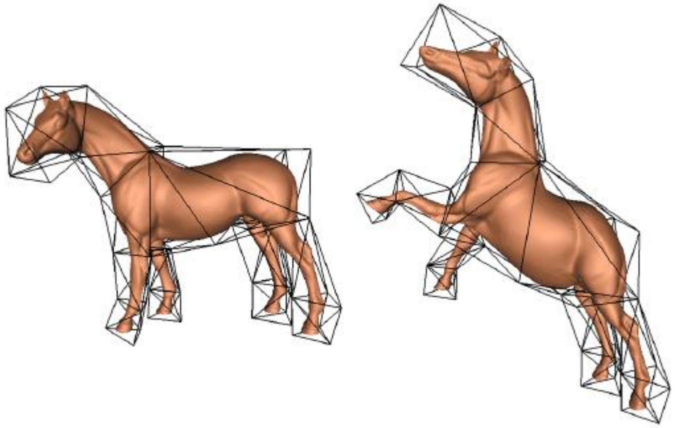


Deformation: Common Paradigms

- **Surface based deformation**
  - Optimization on the surface
  - Physically motivated: variants of elastic energy minimization

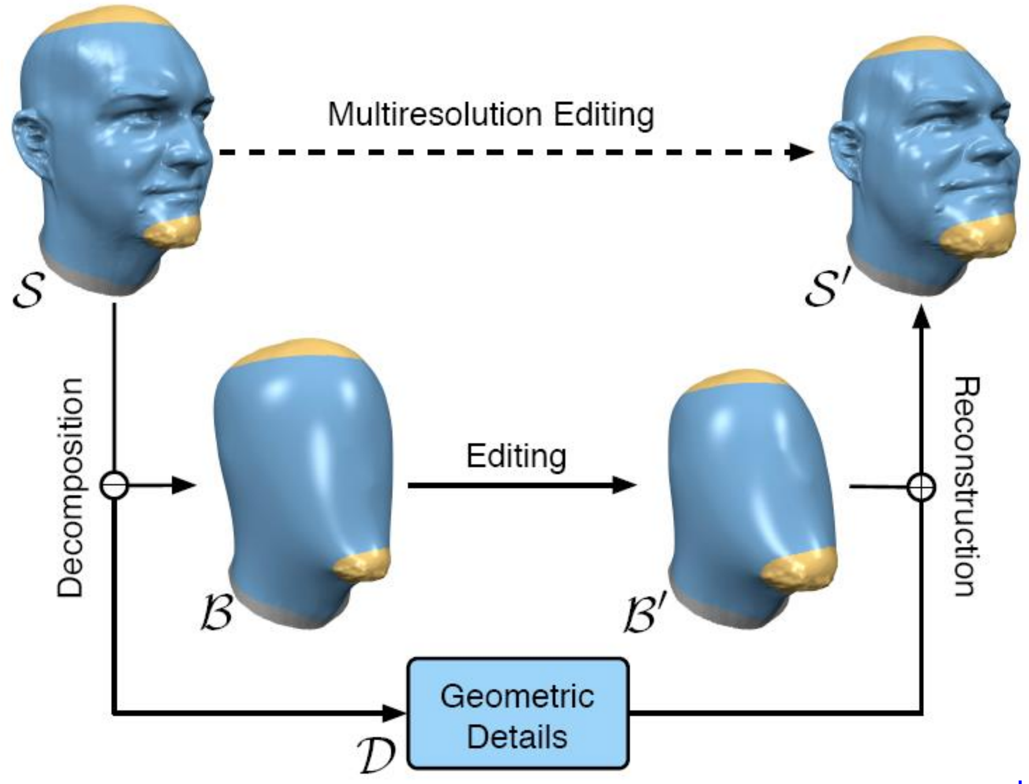


- **Space deformation**
  - Deforms some 2D/3D space using a *cage*
  - Deformation propagation to all points in the space
  - Independent of shape representation



Surface-based Deformations: Example

- Multi-resolution mesh editing



http://igl.ethz.ch/projects/deformation-survey/

Surface-based Deformations: General Framework

- Find a mesh that optimizes some objective functional and satisfies modeling constraints

$$\mathbf{x}_{\text{def}} = \underset{\mathbf{x}'}{\operatorname{argmin}}\, E(\mathbf{x}') \quad s.t. \quad \mathbf{x}'_i = \mathbf{c}_i \ \forall i \in \mathcal{C}$$

■ $\mathcal{C}$

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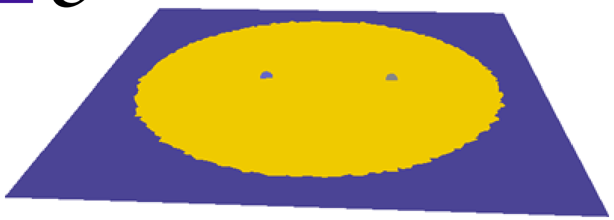

Surface-based Deformations

- Objective functional expressed in the mesh elements (vertices)
- Complexity depends on the mesh resolution
- Linear methods : today and next week
  - Solve a global linear system on the mesh
  - Usually lower quality
- Nonlinear methods : next weeks
  - Higher quality but slower, and harder to implement

Deformation: Common Paradigms

- **Surface based deformation**
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- **Space deformation**
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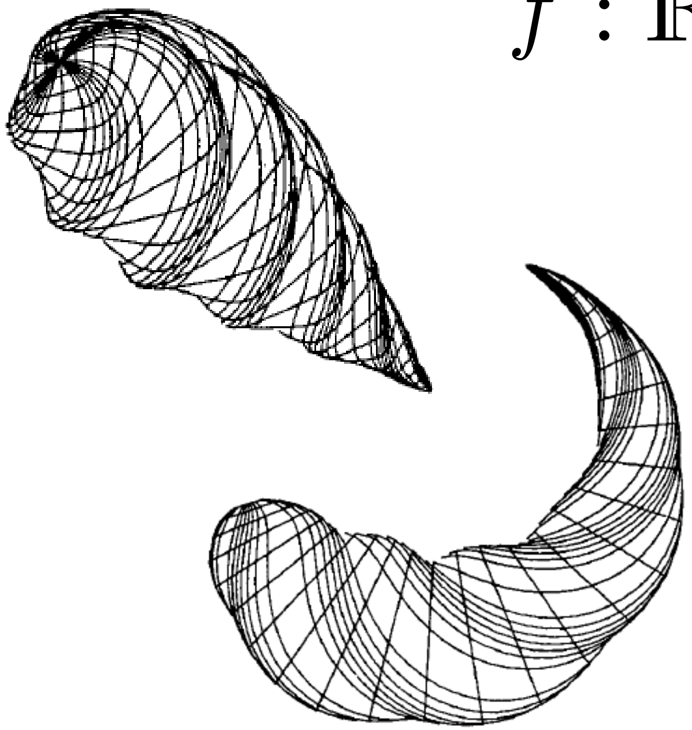
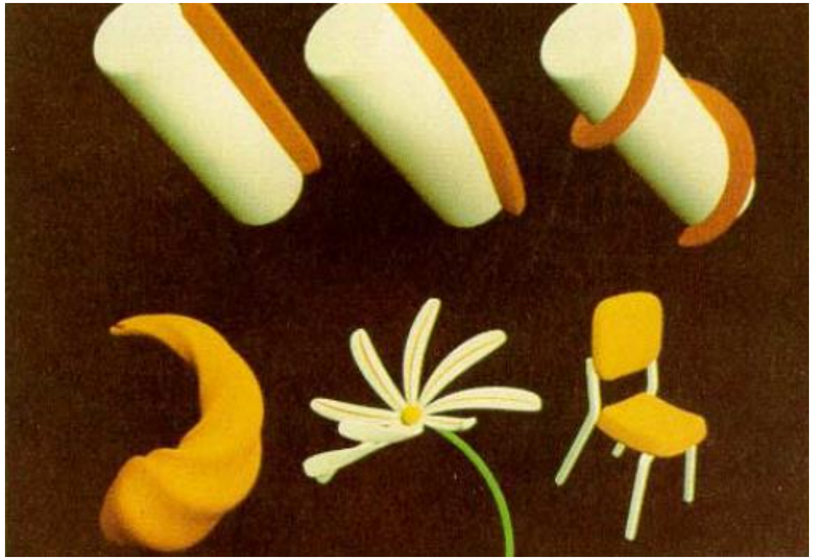
Space Deformations

Early seminal work in computer graphics

- Global and local deformation of solids [Barr 1984]
  http://dl.acm.org/citation.cfm?id=808573

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$



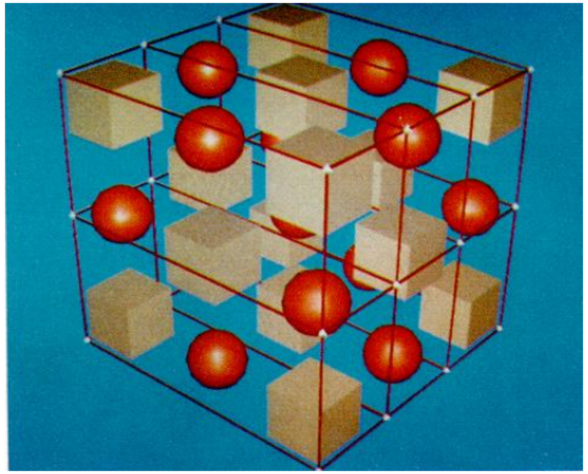


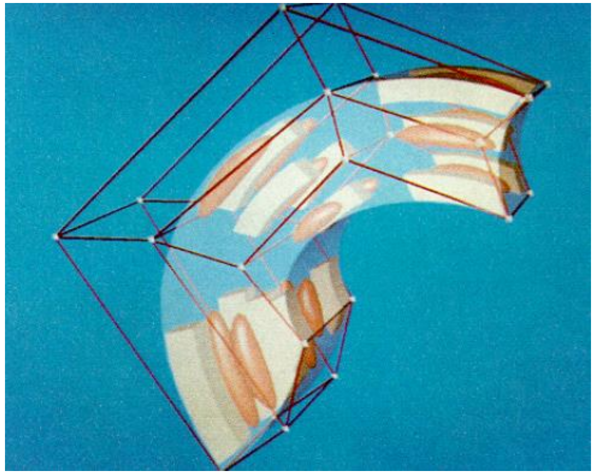
Space Deformations

Early seminal work in computer graphics

- Free form deformations
  [Sederberg and Parry 1986] http://dl.acm.org/citation.cfm?id=15903
  - Uses trivariate tensor product polynomial basis

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$



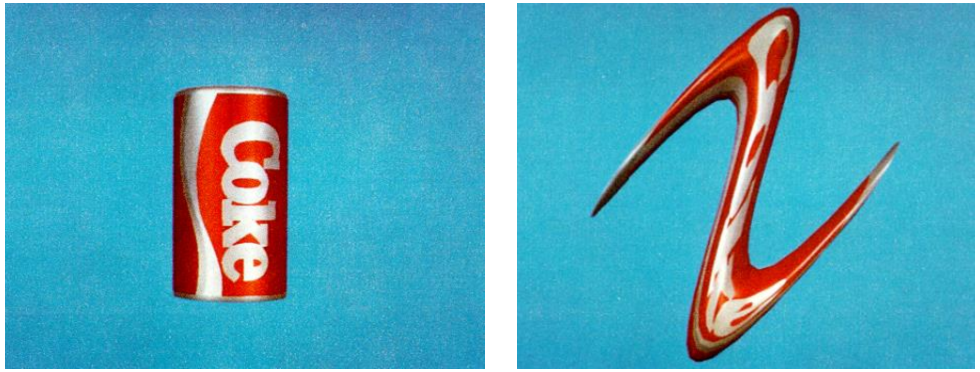


Space Deformations

Early seminal work in computer graphics

- Can be designed to be volume preserving

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

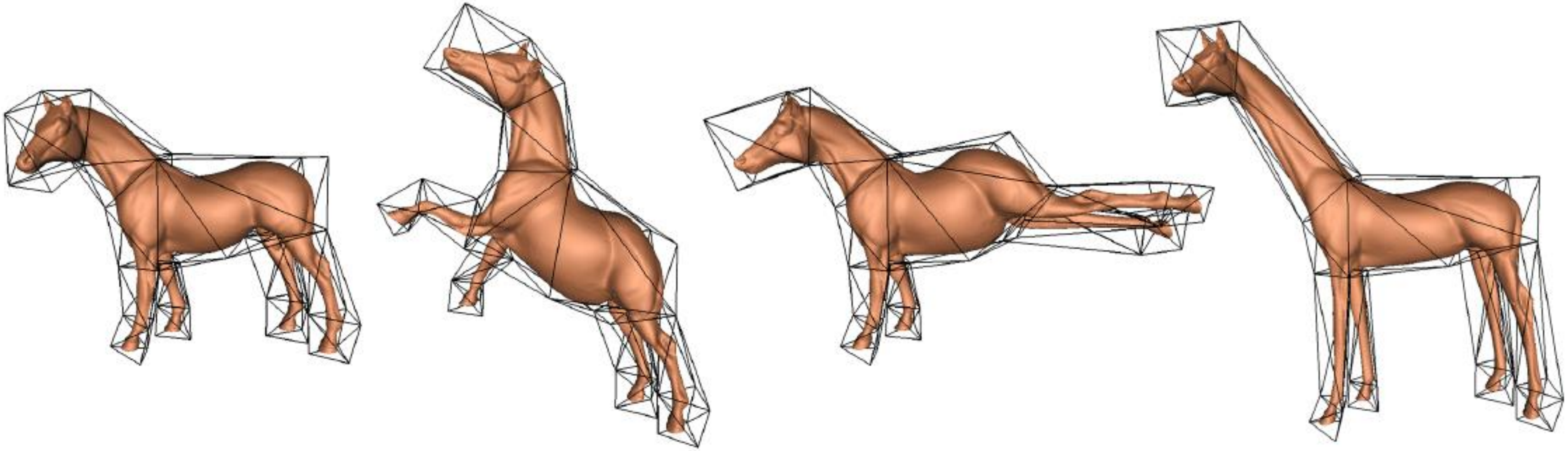


$$\mathbf{F}(x,y,z) = (F(x,y,z), G(x,y,z), H(x,y,z))$$

then the Jacobian is the determinant

$$Jac(\mathbf{F}) = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{vmatrix}$$

More general ‘cages’



Space Deformations: Basic Idea

- Design a set of coordinates for all points in R^d w.r.t. the "cage" vertices
  - Each point $\mathbf{x}$ can be represented as a weighted sum of cage points $\mathbf{p}_i$

$$\mathbf{x} = \sum_{i=1}^{k} w_i(\mathbf{x})\,\mathbf{p}_i$$

  - When the cage changes, the coordinates stay the same, substitute the new cage geometry:

$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\,\mathbf{p}'_i$$

Space Deformations: Basic Idea

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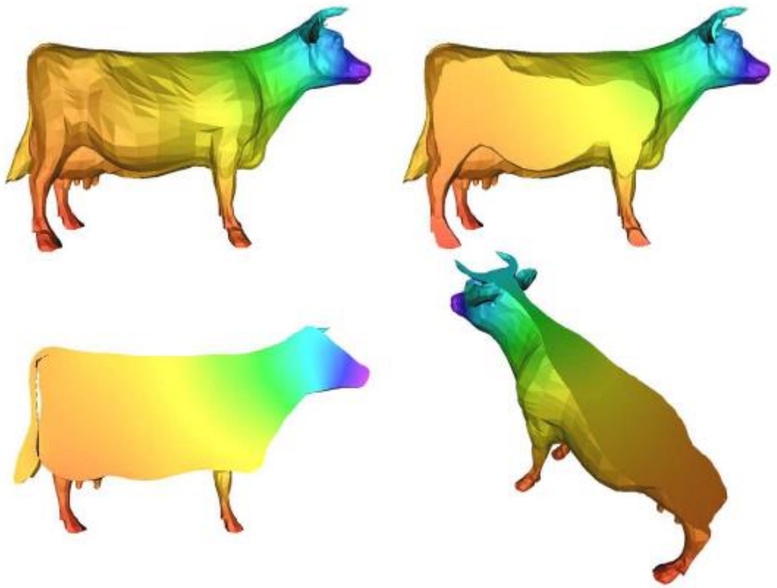
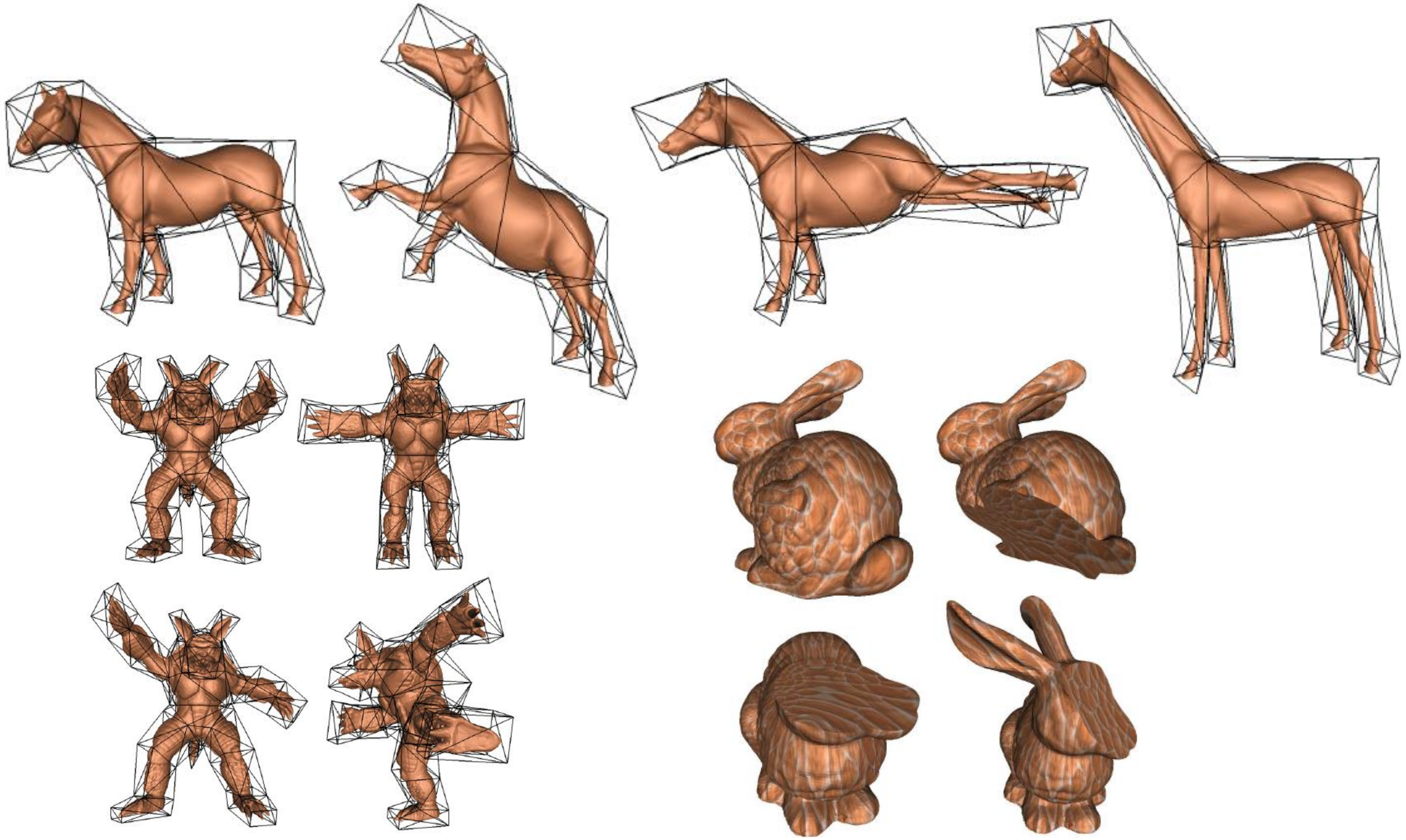


  - The coordinates are smoothly varying and guarantee continuity inside the volume

Space Deformations: Examples

- Mean value coordinates for closed triangle meshes

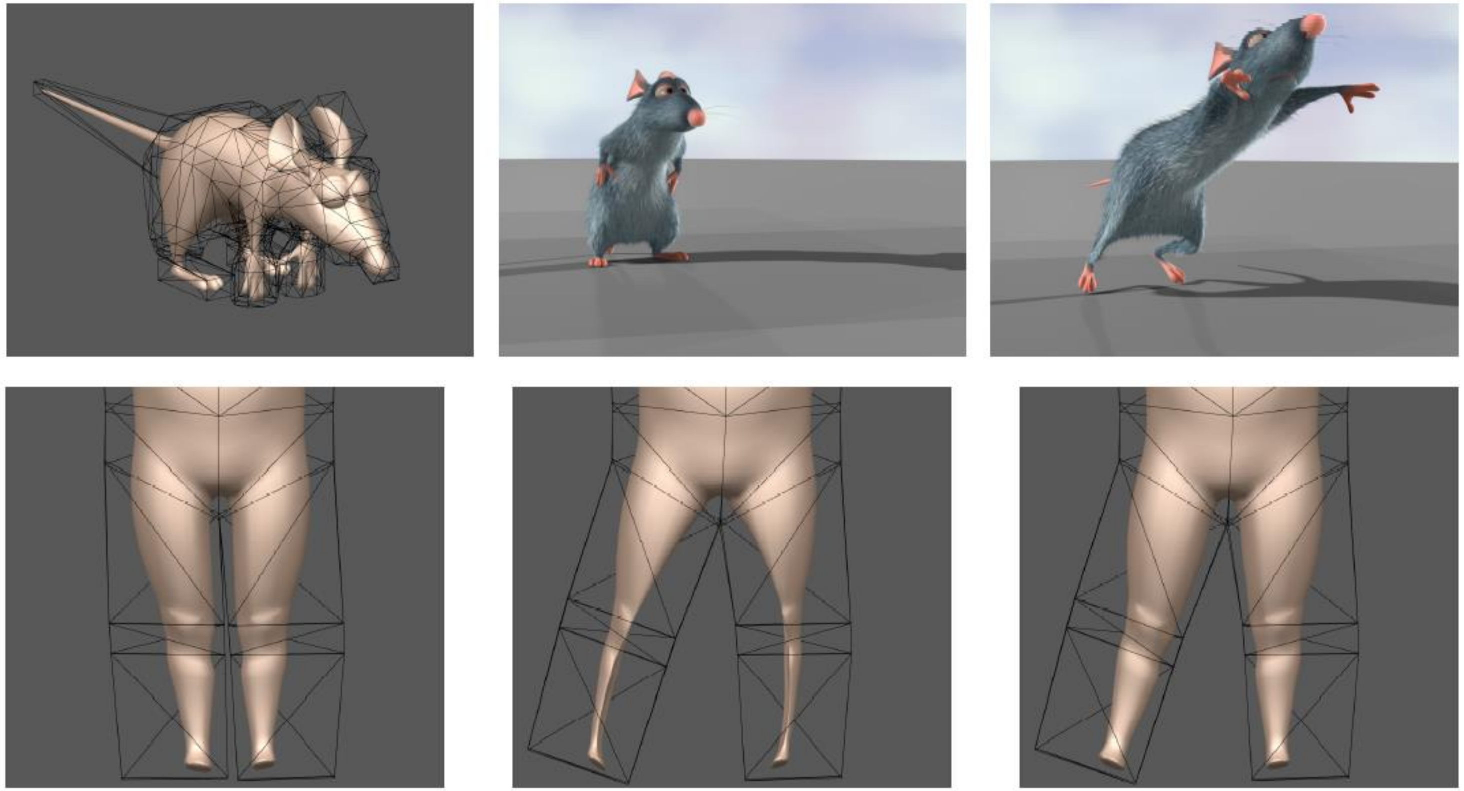


$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\,\mathbf{p}_i'$$

[Ju et al. 2005]
http://dl.acm.org/citation.cfm?id=1186822.1073229

Space Deformations: Examples

- Harmonic coordinates [Joshi et al. 2007]



http://dl.acm.org/citation.cfm?id=1276466

Space Deformations

- Complexity depends mainly on the cage; linear in the number of mesh elements
  - Parallel execution, GPU!
- Can handle disconnected components, non-manifold and non-orientable surfaces or even just point sets
- Harder to control the surface properties since the whole space is being warped

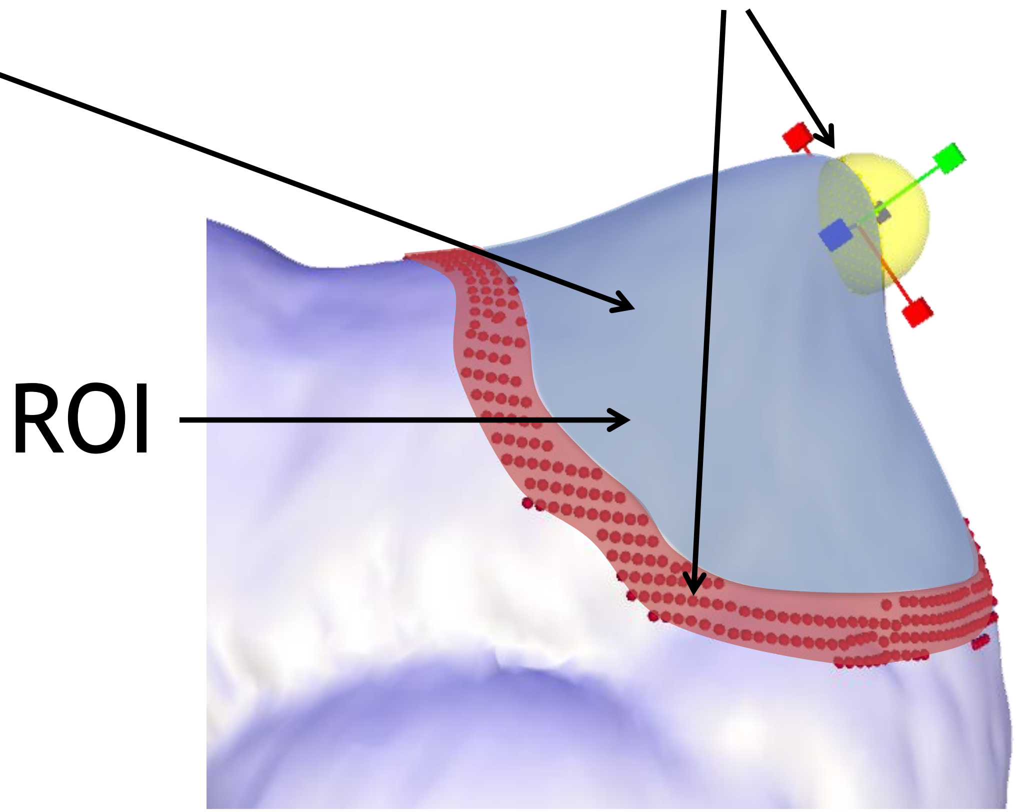
Part I

Surface-based Differential Deformations

Surface-based Deformation: ROI-Handle Editing Metaphor

$$\mathbf{x}_{\text{def}} = \underset{\mathbf{x}'}{\operatorname{argmin}}\, E(\mathbf{x}') \quad s.t. \quad \mathbf{x}'_i = \mathbf{c}_i \;\forall i \in \mathcal{C}$$

- ROI is bounded by a belt (static anchors)
- Manipulation through handle(s) – affine transformations



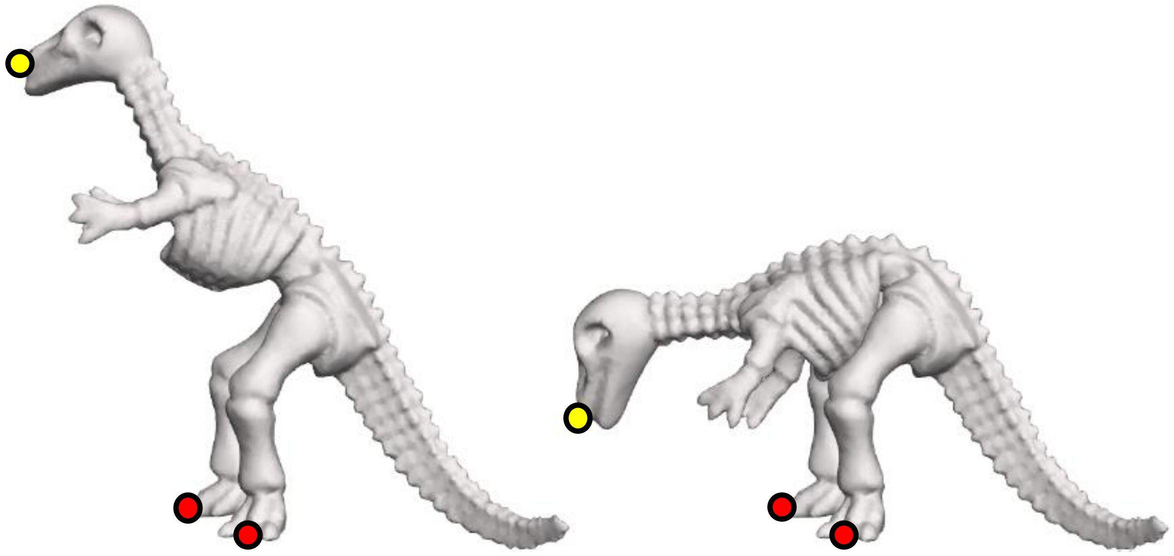
Surface-based Deformation: ROI-Handle Editing Metaphor

Laplacian Mesh Editing

A short editing session
with the *Octopus*

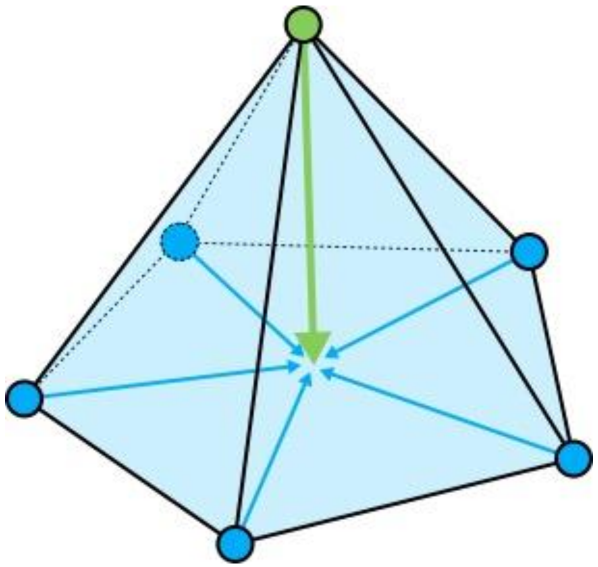
How to Define $E(\mathbf{x}')$?

- Intuitive deformations:
  - Smooth deformation on the global scale
  - Preserve local details (curvatures)
- Invariants: $E(\mathbf{x}')$ should be zero if $\mathbf{x}'$ is a rigid transformation of original geometry $\mathbf{x}$



Recap: Differential Coordinates

- Detail = *smooth*(surface) - surface
- Smoothing = averaging

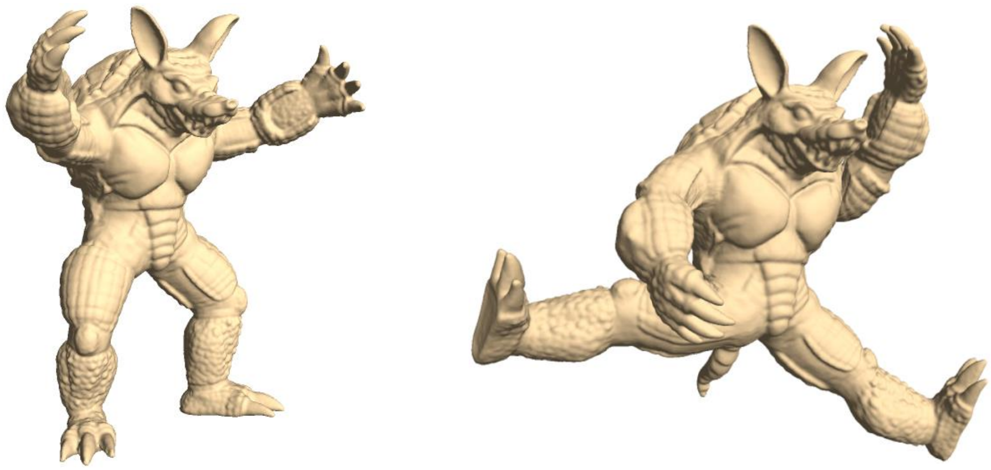


$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij} (\mathbf{x}_j - \mathbf{x}_i)$$

$$W_i = \sum_{j \in \mathcal{N}(i)} w_{ij}$$

Recap: Differential Coordinates

- Represent ***local detail*** at each surface point
  - More descriptive of the shape than just xyz
- Linear transition from xyz to δ
- Useful for operations on surfaces where surface details are important



Simple Laplacian Editing

- Preserve mean curvature normal [≈ differential coordinates] at every point in the ROI [≈ every vertex of the ROI]

continuous: $$E(\mathcal{S}') = \int_{\mathcal{S}'} \|\Delta \mathbf{x}' - \delta\|^2 d\mathbf{x}'$$

discrete: $$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - \delta_i\|^2$$

Simplifying the Laplacian Energy

$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - \delta_i\|^2$$

$$= \mathbf{x}'^{\top} \underline{L^{\top} M L} \mathbf{x}' - 2\mathbf{x}'^{\top} \underline{L^{\top} M} \delta + \text{const}$$

Simplifying the Laplacian Energy

$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - \delta_i\|^2 = \sum_{i=1}^{n} A_i \left(\Delta(\mathbf{x}'_i)^\top \Delta(\mathbf{x}'_i) - 2\Delta(\mathbf{x}'_i)^\top \delta_i + \delta_i^\top \delta_i\right)$$

$$= \mathbf{x}'^\top \underline{L^\top M L}\mathbf{x}' - 2\mathbf{x}'^\top \underline{L^\top M}\delta + \text{const}$$

$$L^\top M L$$

Symmetric sparse matrix!

Minimizing the Laplacian Energy

$$E(\mathbf{x}') = \mathbf{x}'^{\mathsf{T}} L_w M^{-1} L_w \mathbf{x}' - 2\mathbf{x}'^{\mathsf{T}} L_w \delta + \text{const}$$

$$\frac{\partial}{\partial \mathbf{x}'} E(\mathbf{x}') = \underbrace{2L_w M^{-1} L_w}_{A} \mathbf{x}' - \underbrace{2L_w \delta}_{b}$$

- To find the minimum, set gradient = 0 and substitute the modeling constraints $\mathbf{x}'_i = \mathbf{c}_i, \; i \in \mathcal{C}$

Minimizing the Laplacian Energy

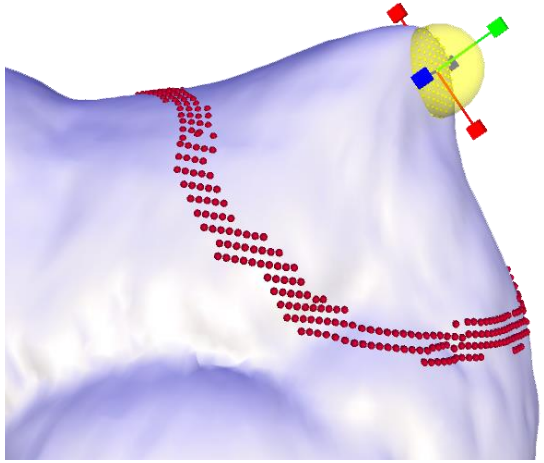
$$\frac{\partial}{\partial \mathbf{x}'} E(\mathbf{x}') = \boxed{2L_w M^{-1} L_u} \mathbf{x}' - \boxed{2L_w \delta}$$

A b

$$A\mathbf{x}' = \mathbf{b}$$

Matrix depends on the initial mesh and the indices of the constraints only. Matrix is fixed!

Right-hand side contains the coordinates of the constraints (handles)



Minimizing the Laplacian Energy

$$\frac{\partial}{\partial \mathbf{x}'} E(\mathbf{x}') = \boxed{2L_w M^{-1} L_u} \mathbf{x}' - \boxed{2L_w \delta}$$

Symmetric sparse matrix!

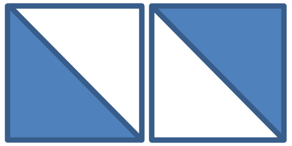
A

b

$$A\mathbf{x}' = \mathbf{b}$$

Sparse Cholesky decomposition:

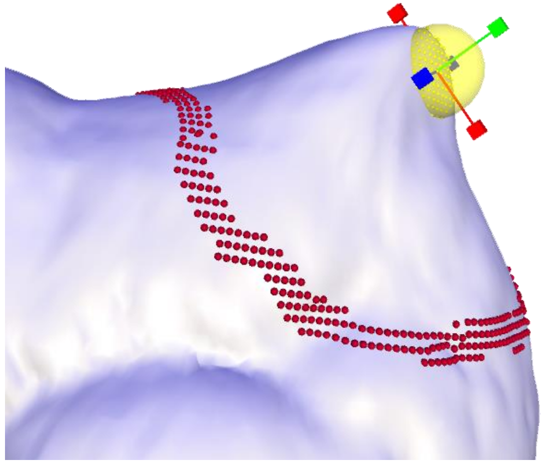
$$A = L_{\text{chol}} L_{\text{chol}}^{\mathsf{T}}$$



At run-time: just back-substitution!

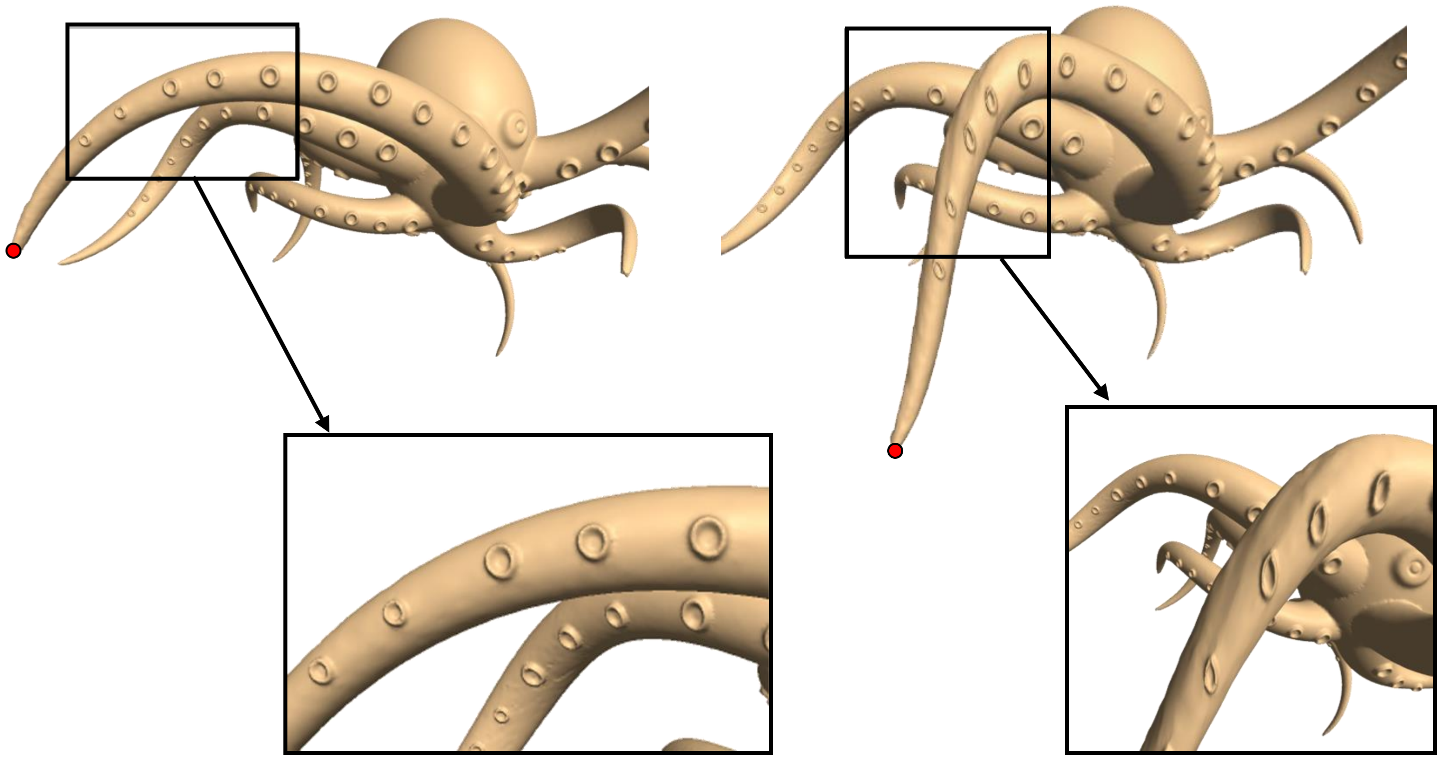
$$L_{\text{chol}} \mathbf{y} = \mathbf{b}$$

$$L_{\text{chol}}^{\mathsf{T}} \mathbf{x}' = \mathbf{y}$$



Fundamental Problem: Invariance to Transformations

- The basic Laplacian operator is ***translation*** -invariant, but not **rotation-**invariant
- E(**x**’) attempts to preserve the **original global** orientation of the details (the normal directions)



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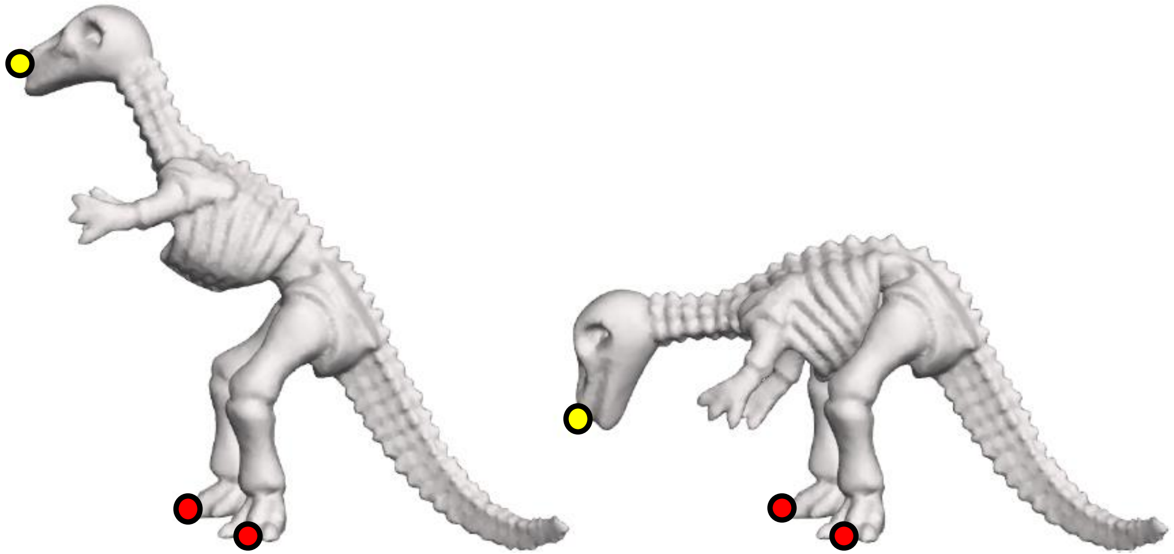

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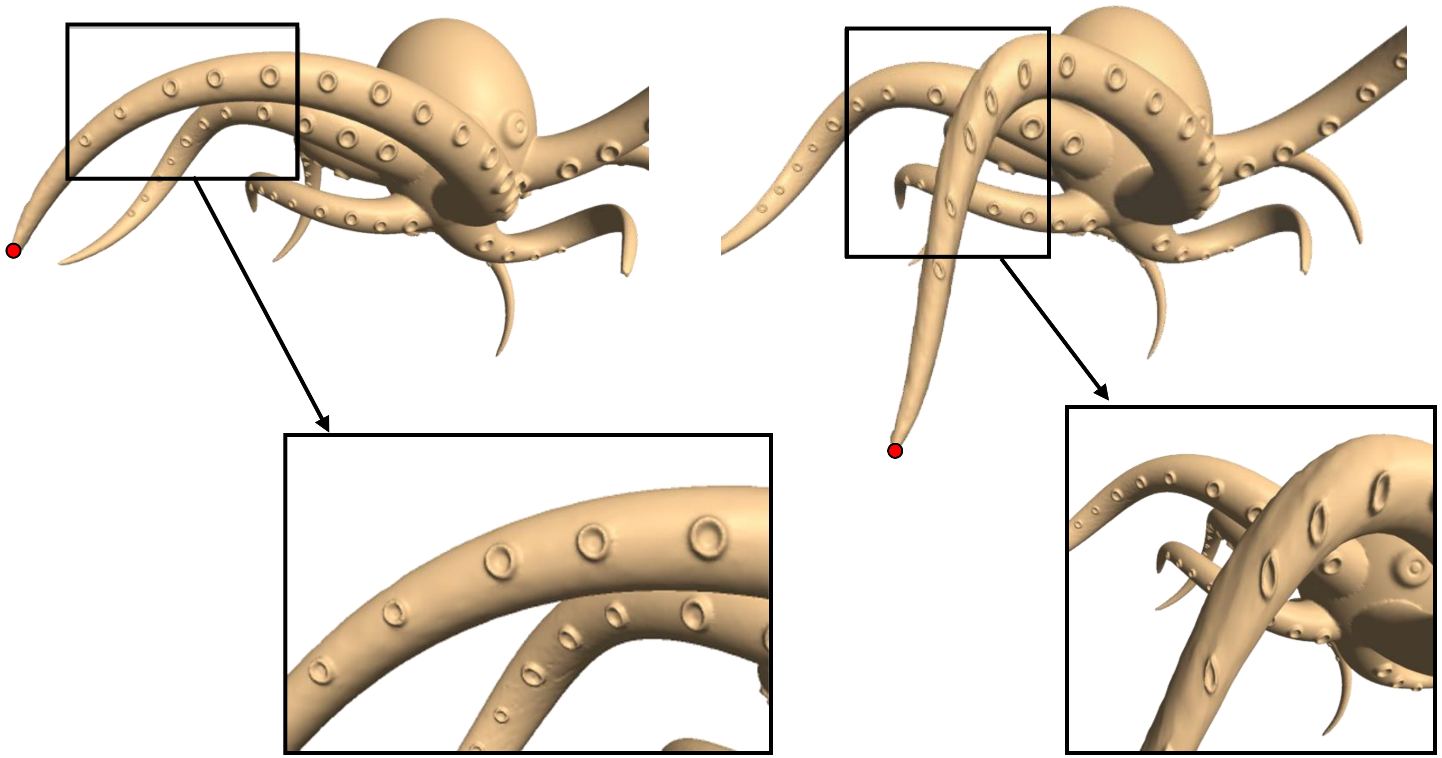

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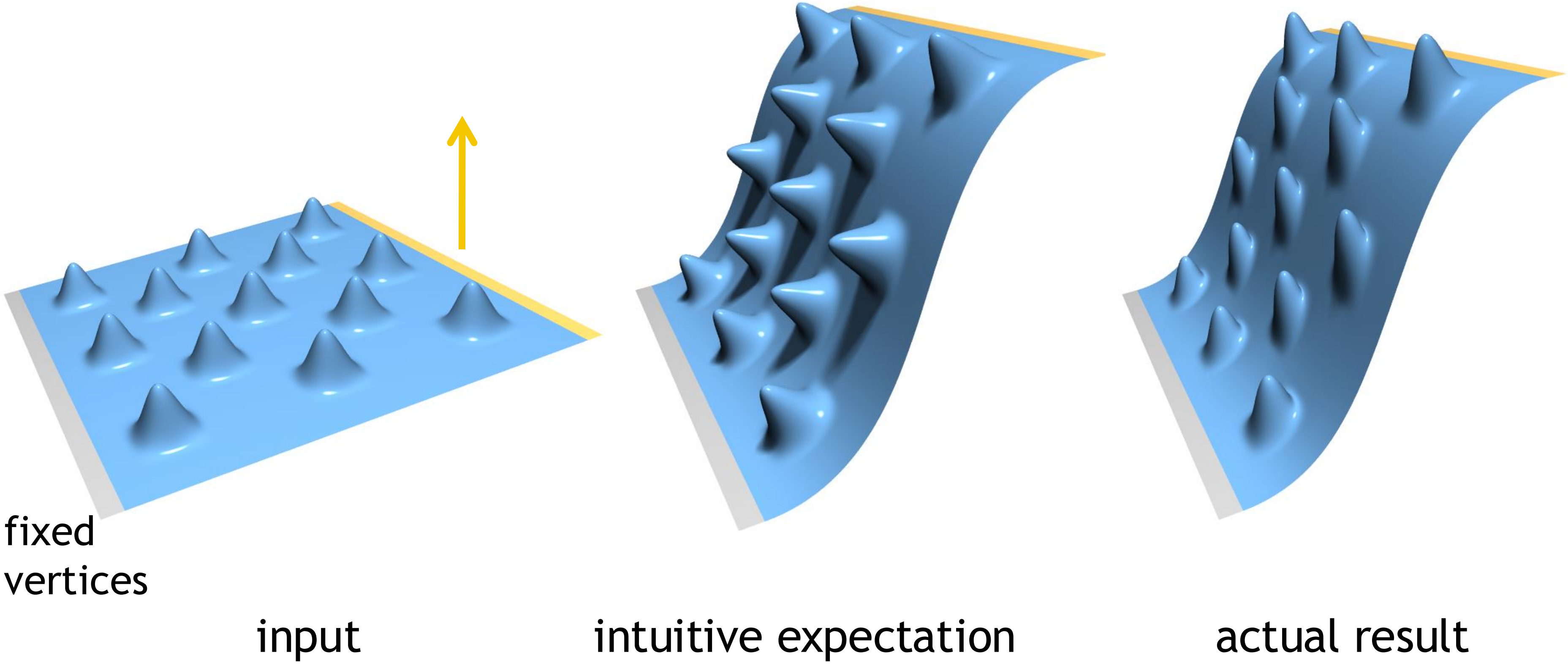

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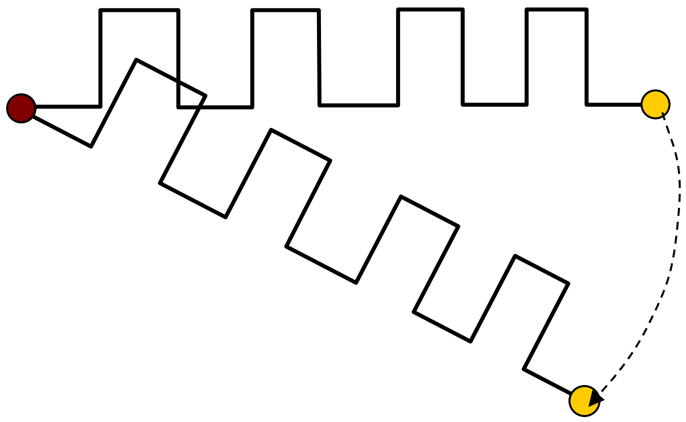

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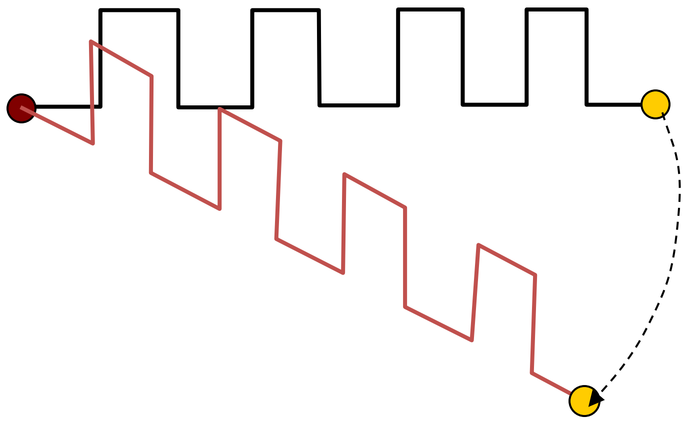


Fundamental Problem: Invariance to Transformations

- The Great Wall of China...


Energy Functional

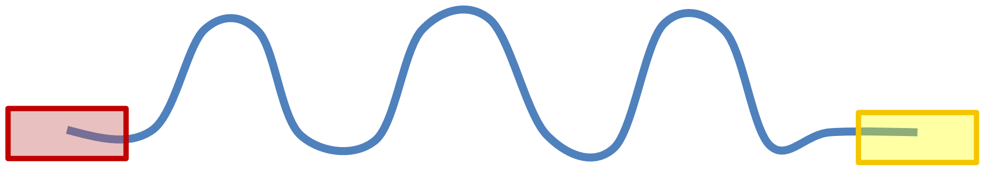
- We need a rigid-invariant energy…

$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - \delta_i\|^2$$

Need to locally rotate the *target* mean-curvature normals

Fixing Local Rotations: Multiresolution Approach

input



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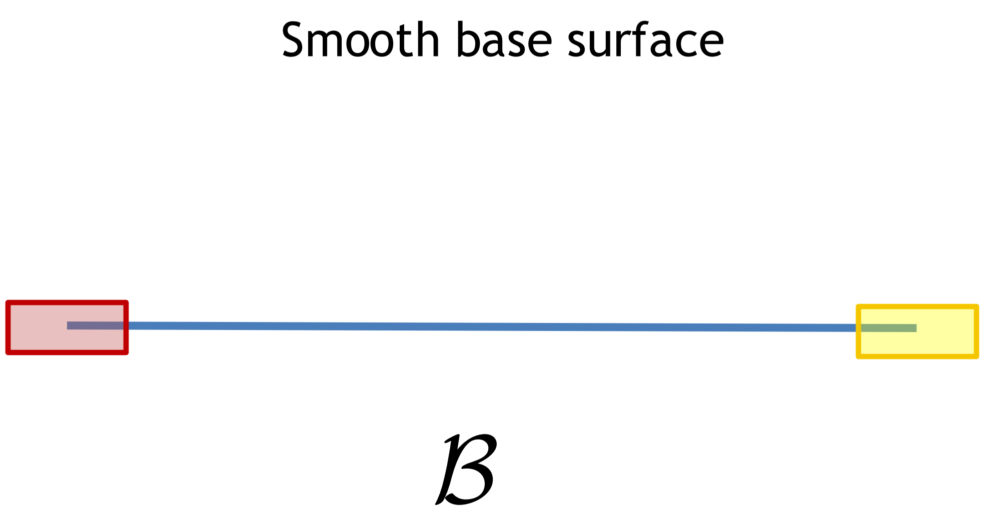
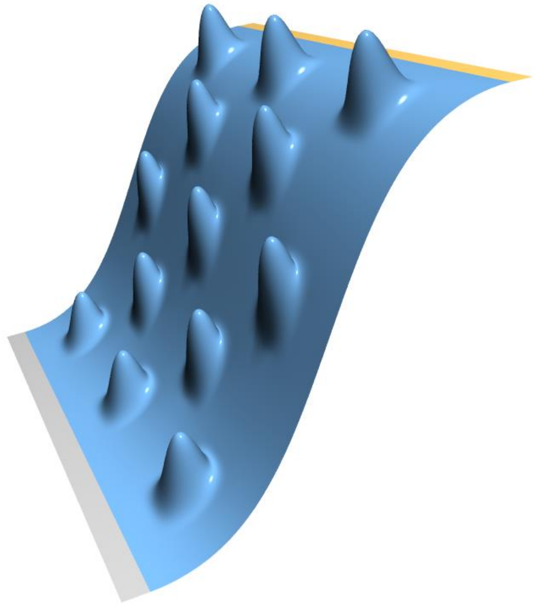
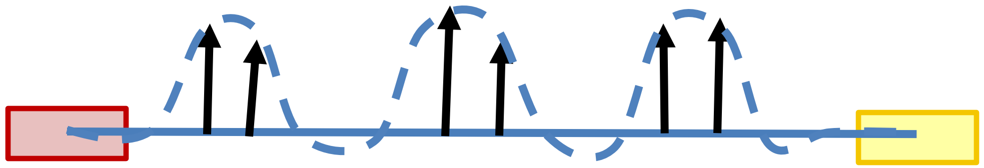
Fixing Local Rotations: Multiresolution Approach





Fixing Local Rotations: Multiresolution Approach

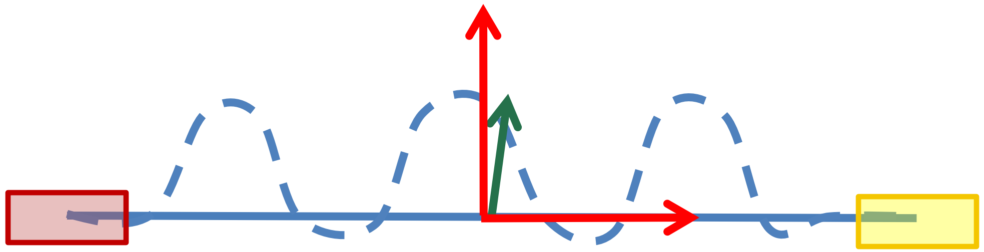
Details – displacement vectors



$$\mathcal{S} - \mathcal{B}$$

Fixing Local Rotations: Multiresolution Approach

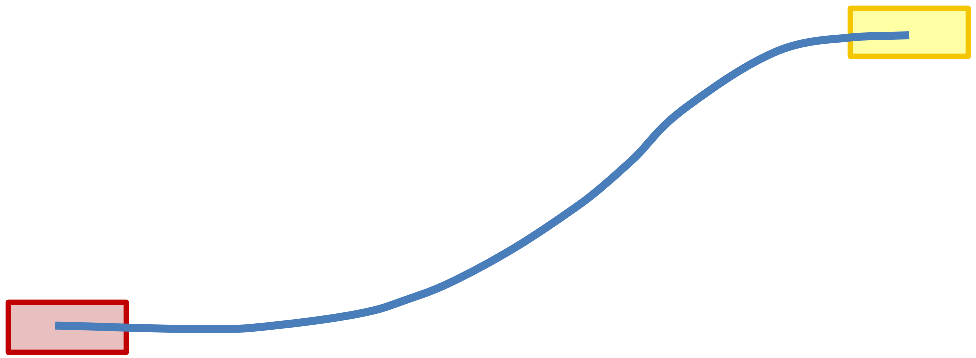
Encode details in the local frame of B



$$\mathbf{d}_i = a_1 \mathbf{t}_i + a_2 \mathbf{n}_i$$

Fixing Local Rotations: Multiresolution Approach

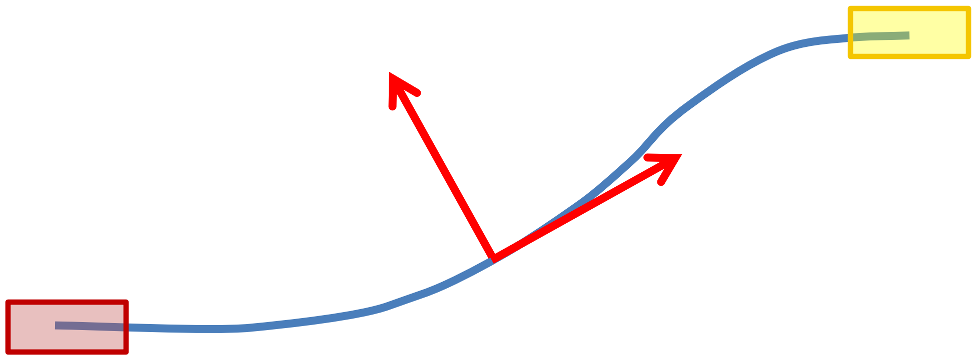
Deform smooth base surface



$\mathcal{B}'$

Fixing Local Rotations: Multiresolution Approach

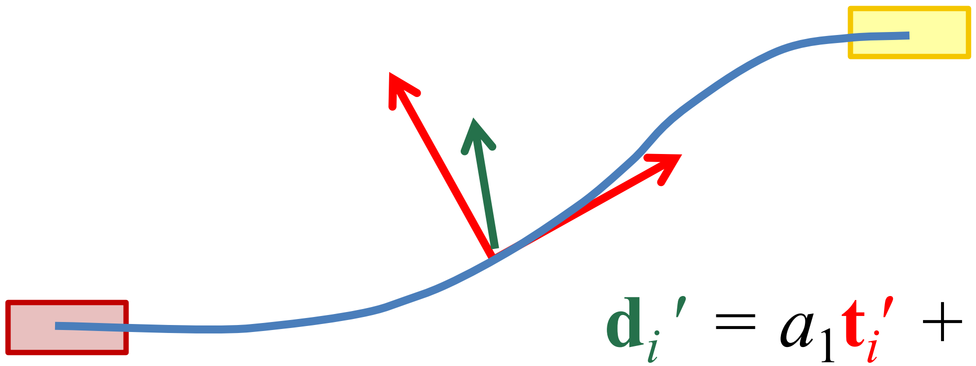
Local frames on B'



$\mathcal{B}'$

Fixing Local Rotations: Multiresolution Approach

Add details back – in local frame!

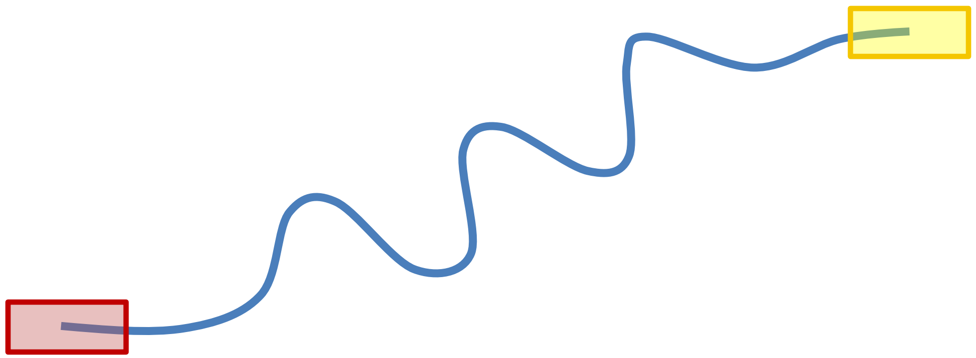


$$\mathbf{d}_i' = a_1\mathbf{t}_i' + a_2\mathbf{n}_i'$$

$\mathcal{B}'$

Fixing Local Rotations: Multiresolution Approach

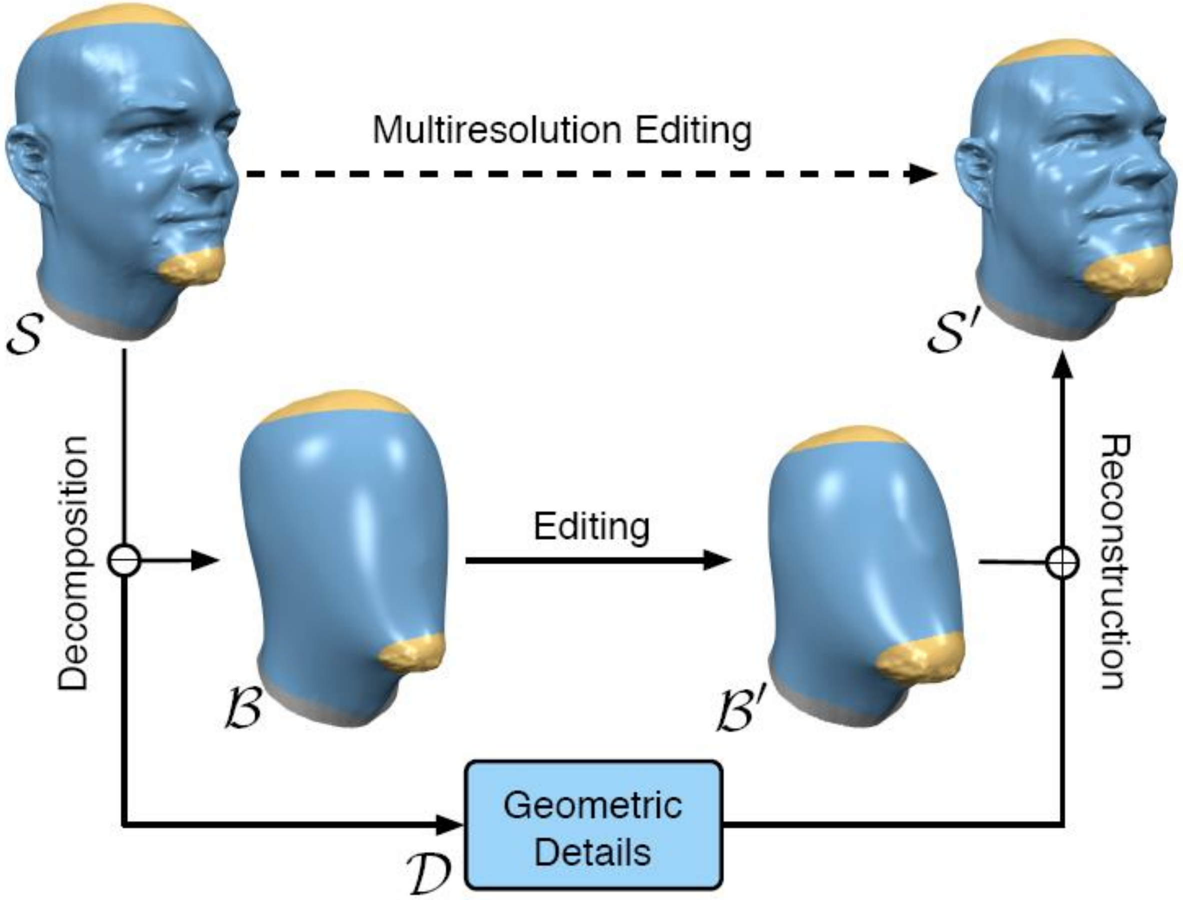
Displace the vertices to get the result



$\mathcal{S}'$

Fixing Local Rotations: Multiresolution Approach

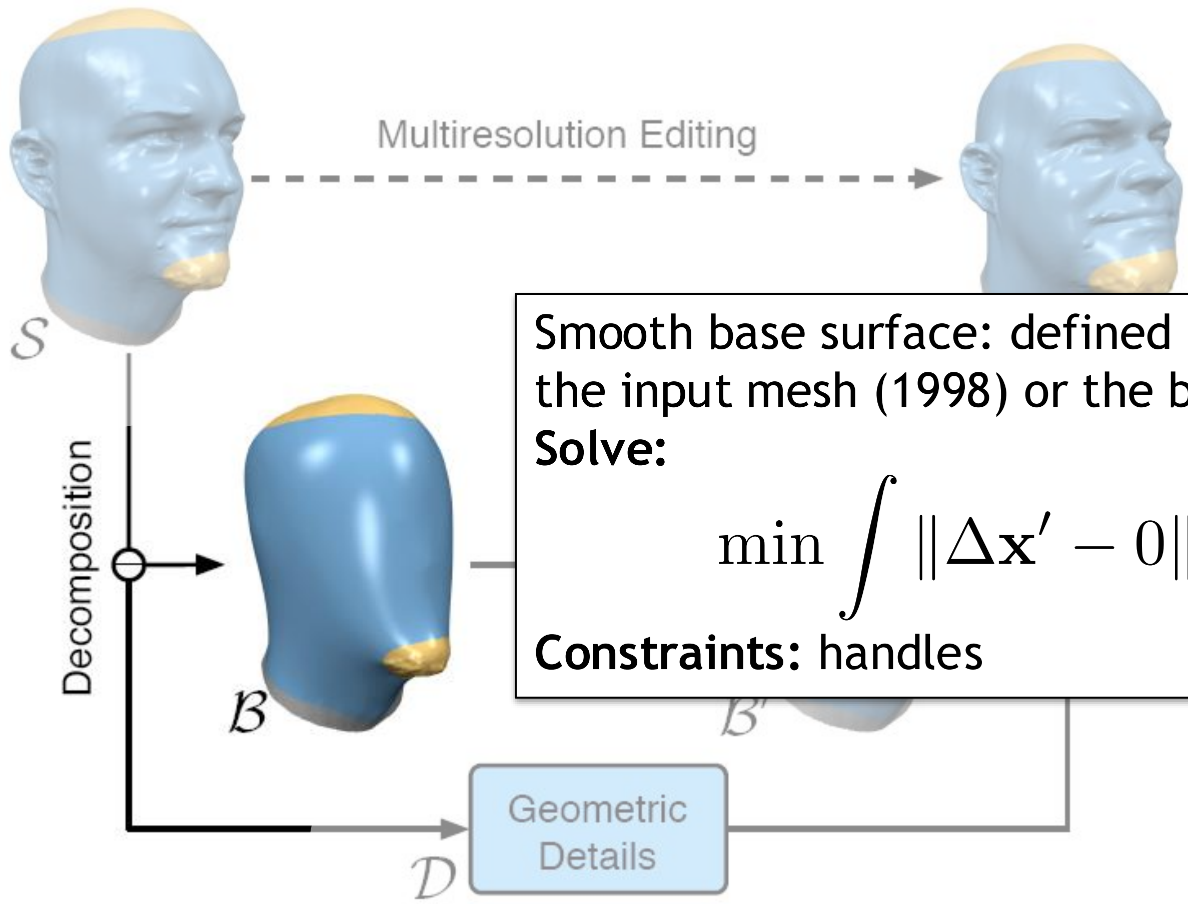
- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004



http://dl.acm.org/citation.cfm?id=280831 dl.acm.org/citation.cfm?id=1015772

Fixing Local Rotations: Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004



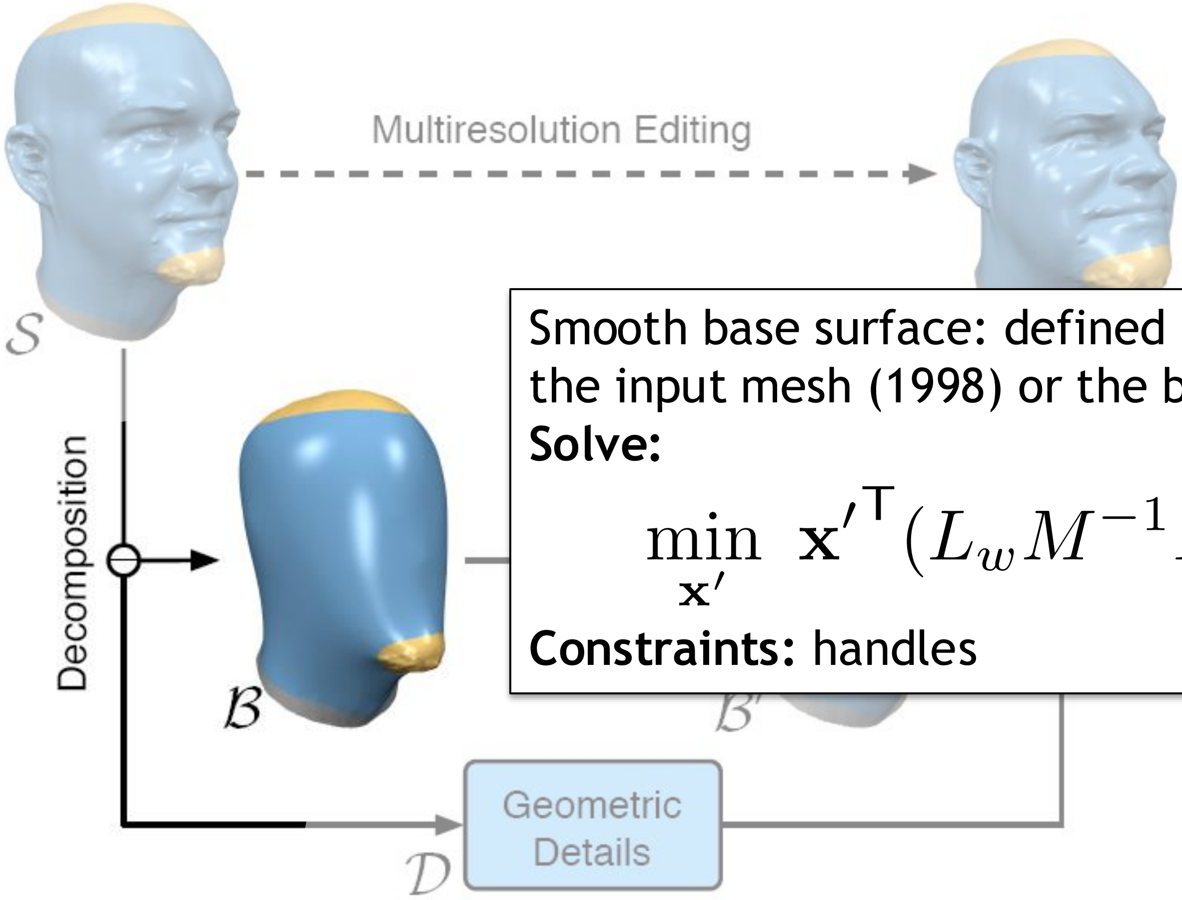
Smooth base surface: defined by Laplacian smoothing of the input mesh (1998) or the bi-Laplacian surface (2004):
Solve:

$$\min \int \|\Delta \mathbf{x}' - 0\|^2 d\mathbf{x}'$$

Constraints: handles

Fixing Local Rotations: Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004



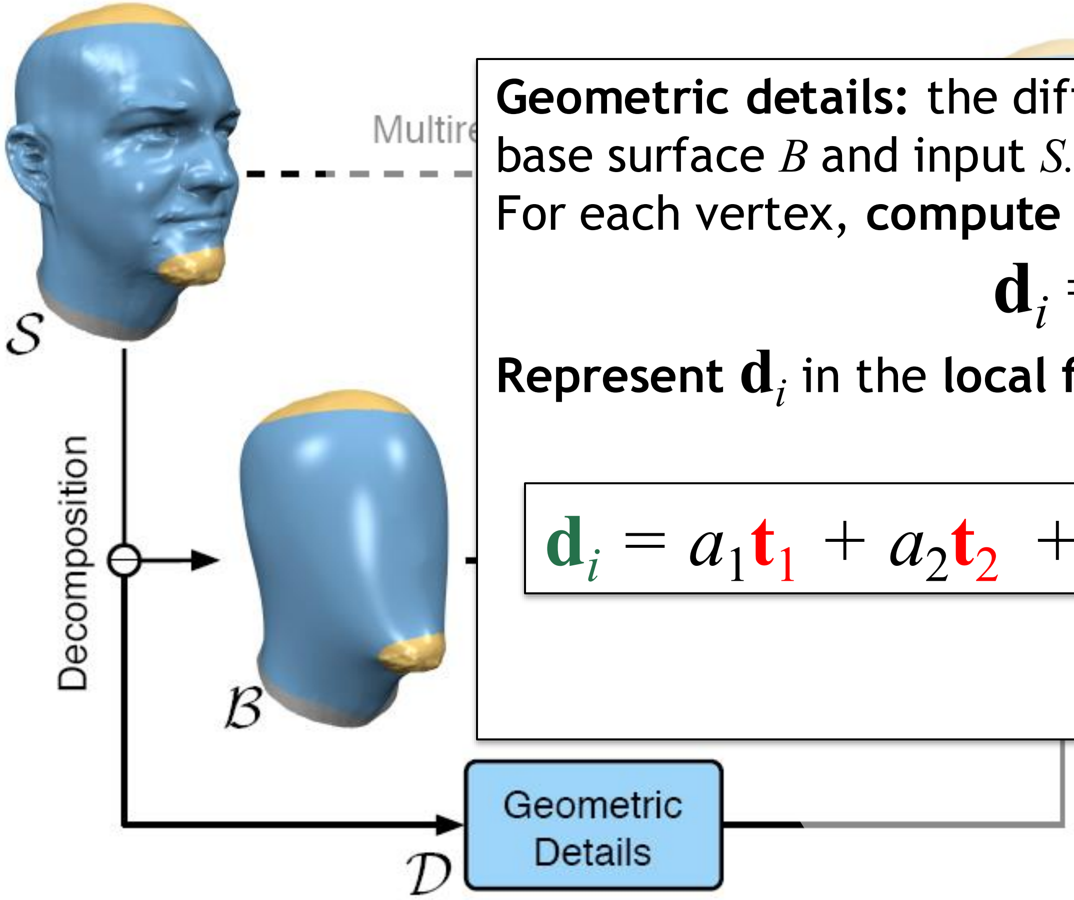
Smooth base surface: defined by Laplacian smoothing of the input mesh (1998) or the bi-Laplacian surface (2004):
Solve:

$$\min_{\mathbf{x}'} \mathbf{x}'^{\mathsf{T}} (L_w M^{-1} L_w)\, \mathbf{x}'$$

Constraints: handles

Fixing Local Rotations: Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004

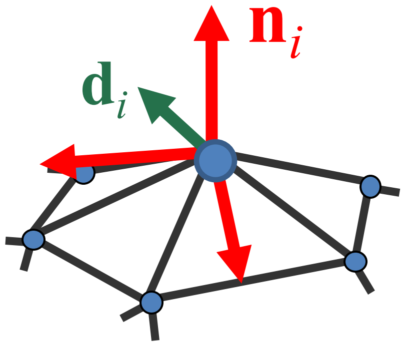


Geometric details: the difference vectors between base surface B and input S.
For each vertex, **compute**

$$\mathbf{d}_i = \mathbf{v}_i - \mathbf{b}_i$$

Represent $\mathbf{d}_i$ in the **local frame** of $\mathbf{b}_i$

$$\mathbf{d}_i = a_1\mathbf{t}_1 + a_2\mathbf{t}_2 + a_3\mathbf{n}_i$$



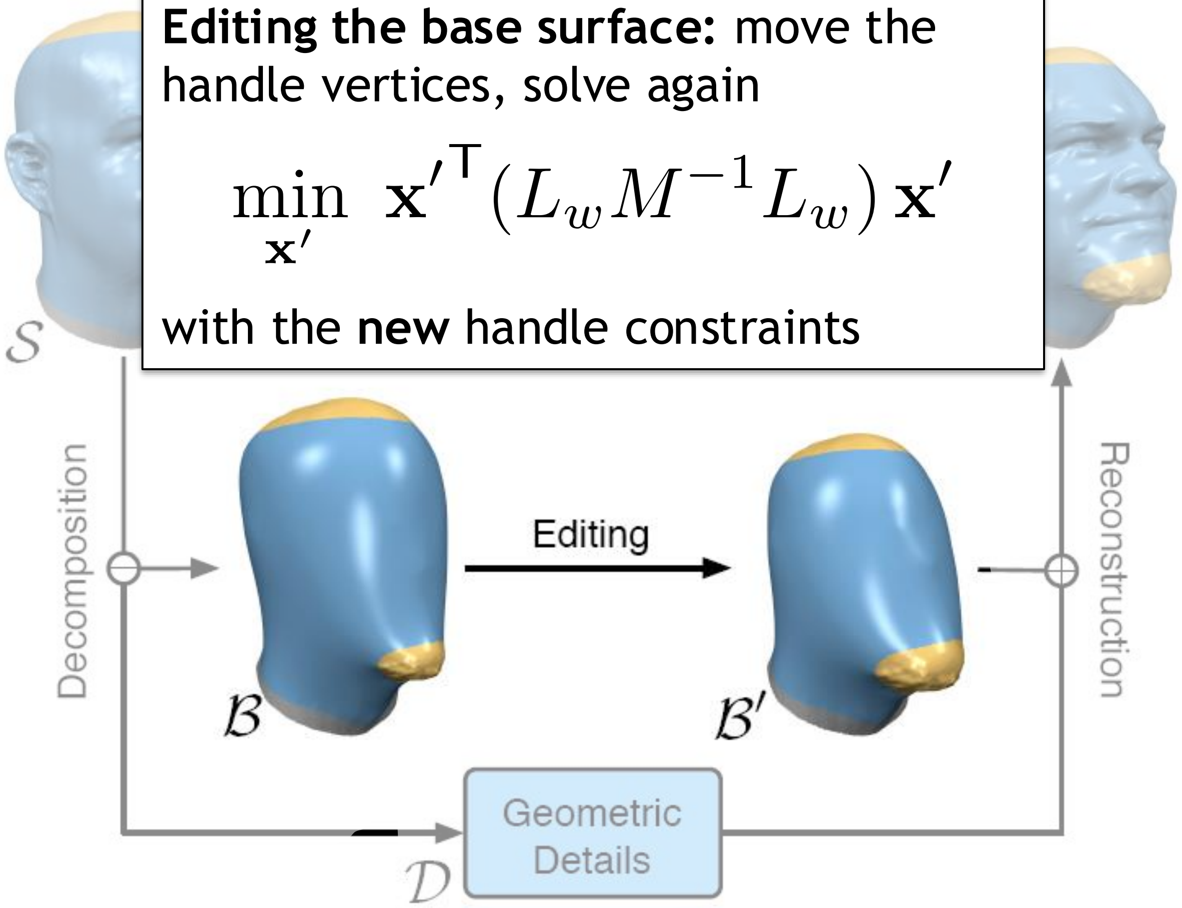
Fixing Local Rotations: Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004

Editing the base surface: move the handle vertices, solve again

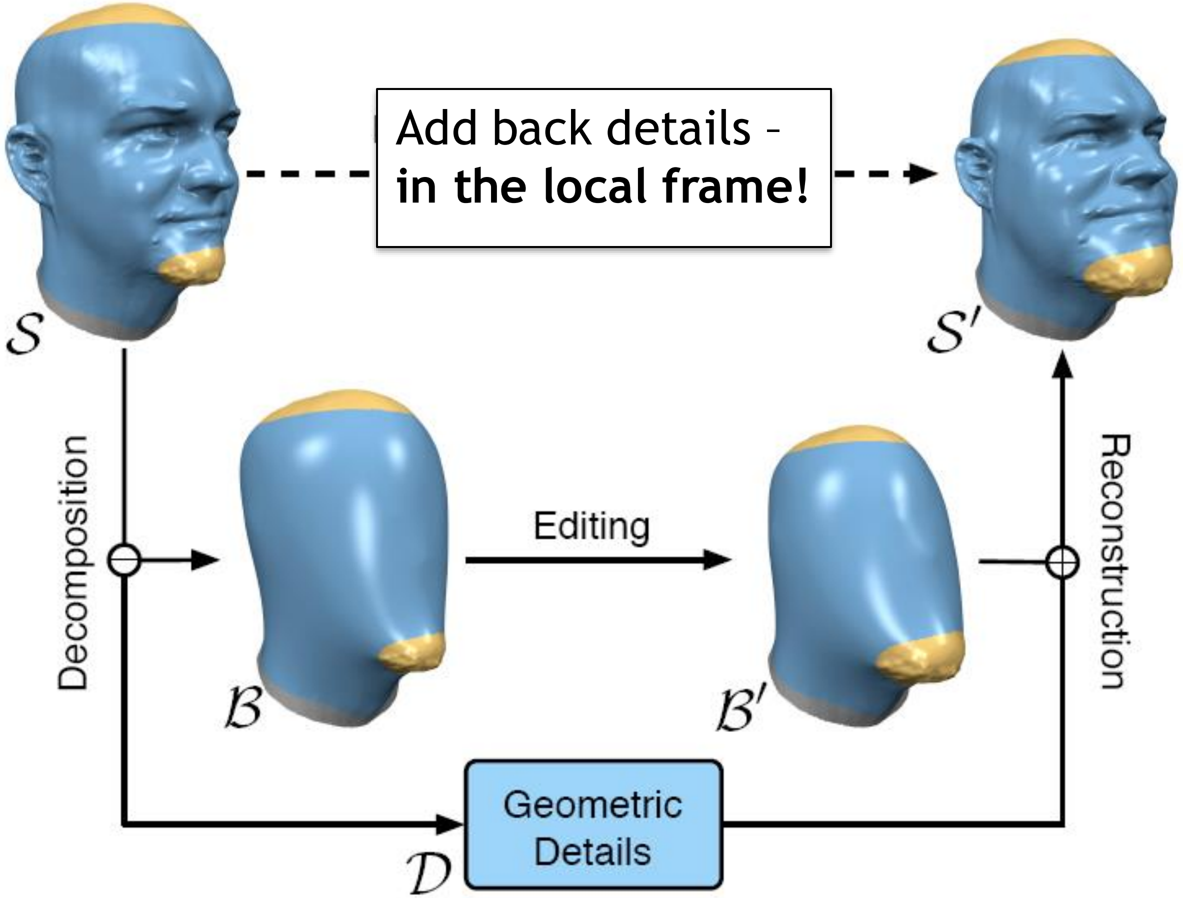
$$\min_{\mathbf{x}'} \ \mathbf{x}'^{\mathsf{T}} (L_w M^{-1} L_w)\, \mathbf{x}'$$

with the **new** handle constraints



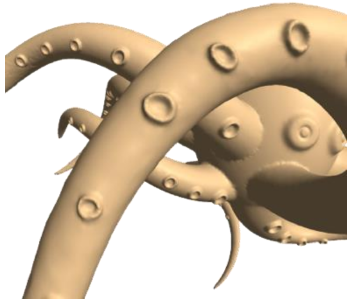
Fixing Local Rotations: Multiresolution Approach

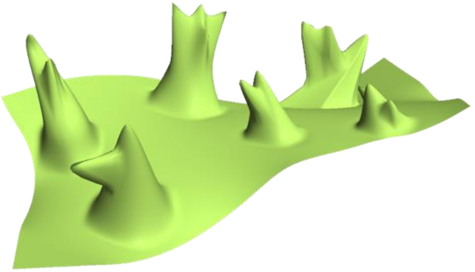
- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004



Multiresolution Framework: Discussion

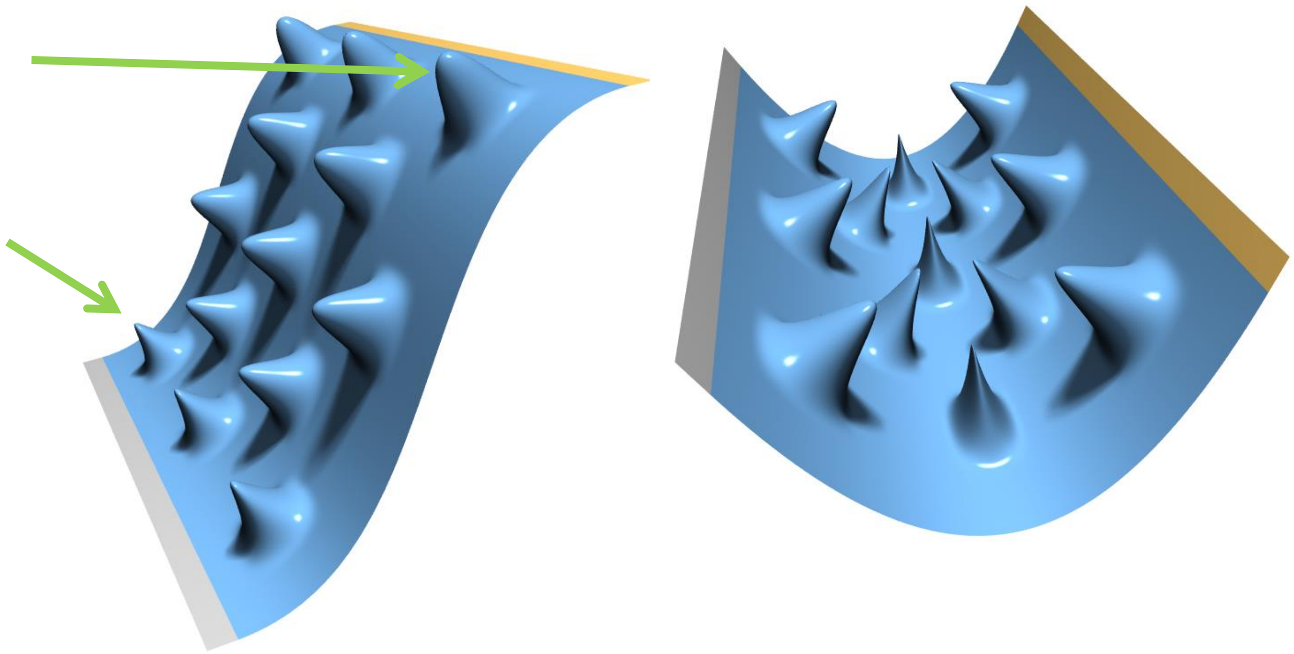
- Advantages:
  - Fast! Linear solve for the base surface deformation, and then add back displacements
  - Intuitive, easy to implement
- Problem: works only for *small* height fields (when detail vectors are small)


almost a height field


not a height field

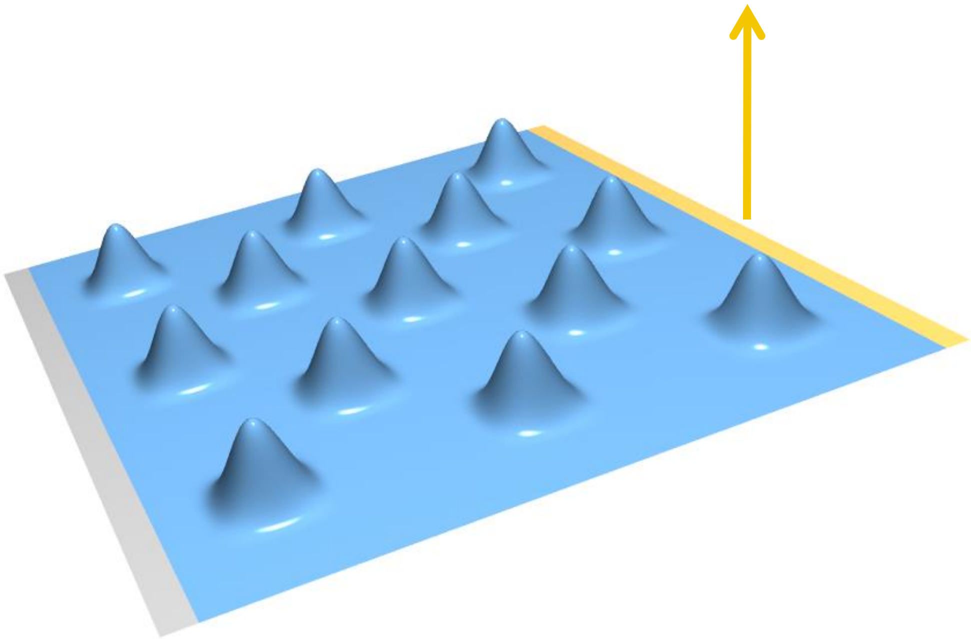
Multiresolution Framework: Discussion

- Problem: If detail vectors are too big, we get overshooting and **self-intersections**, especially in concave cases



Local Rotations – Single Resolution Solutions

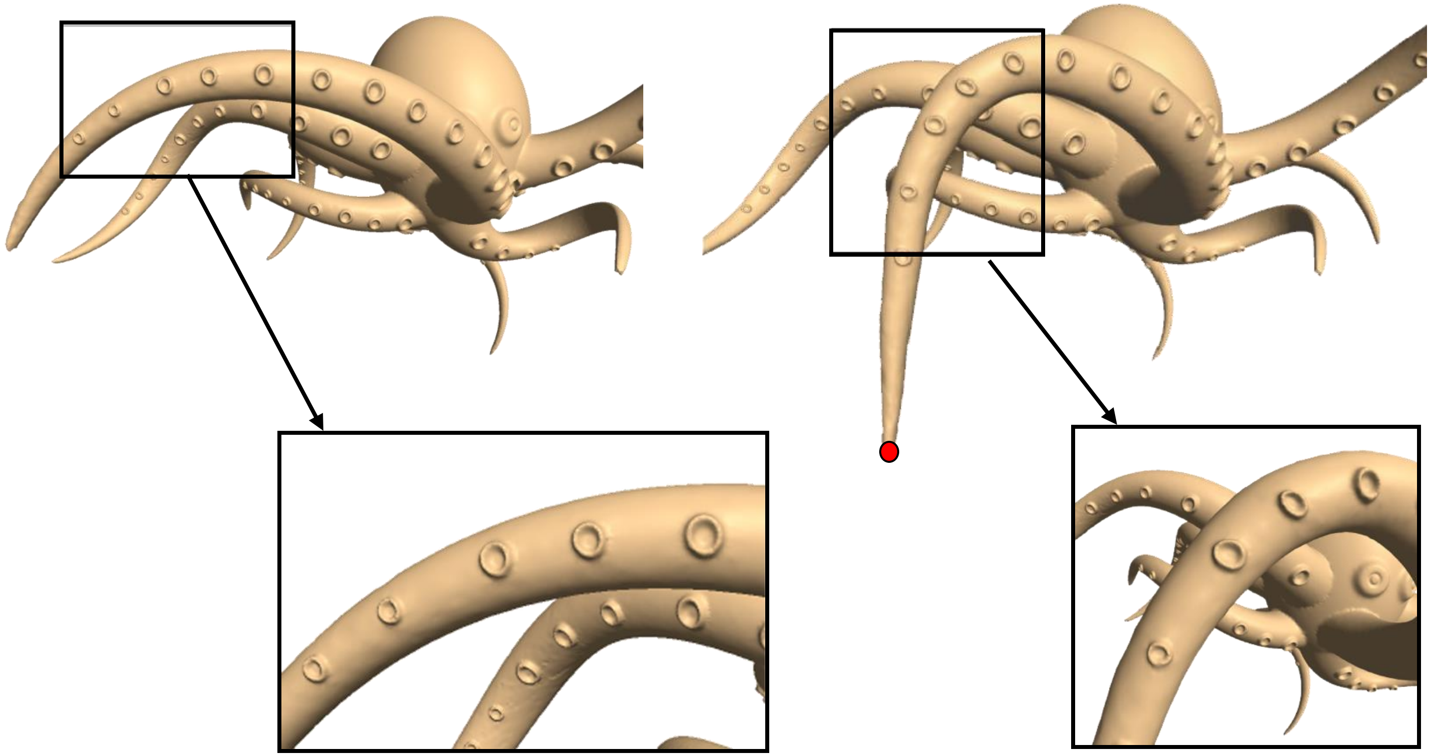
- Come up with a rotation field on the surface based on the modeling constraints
- Rotate the differential coordinates; solve



Estimation of Rotations

Lipman et al. 2004

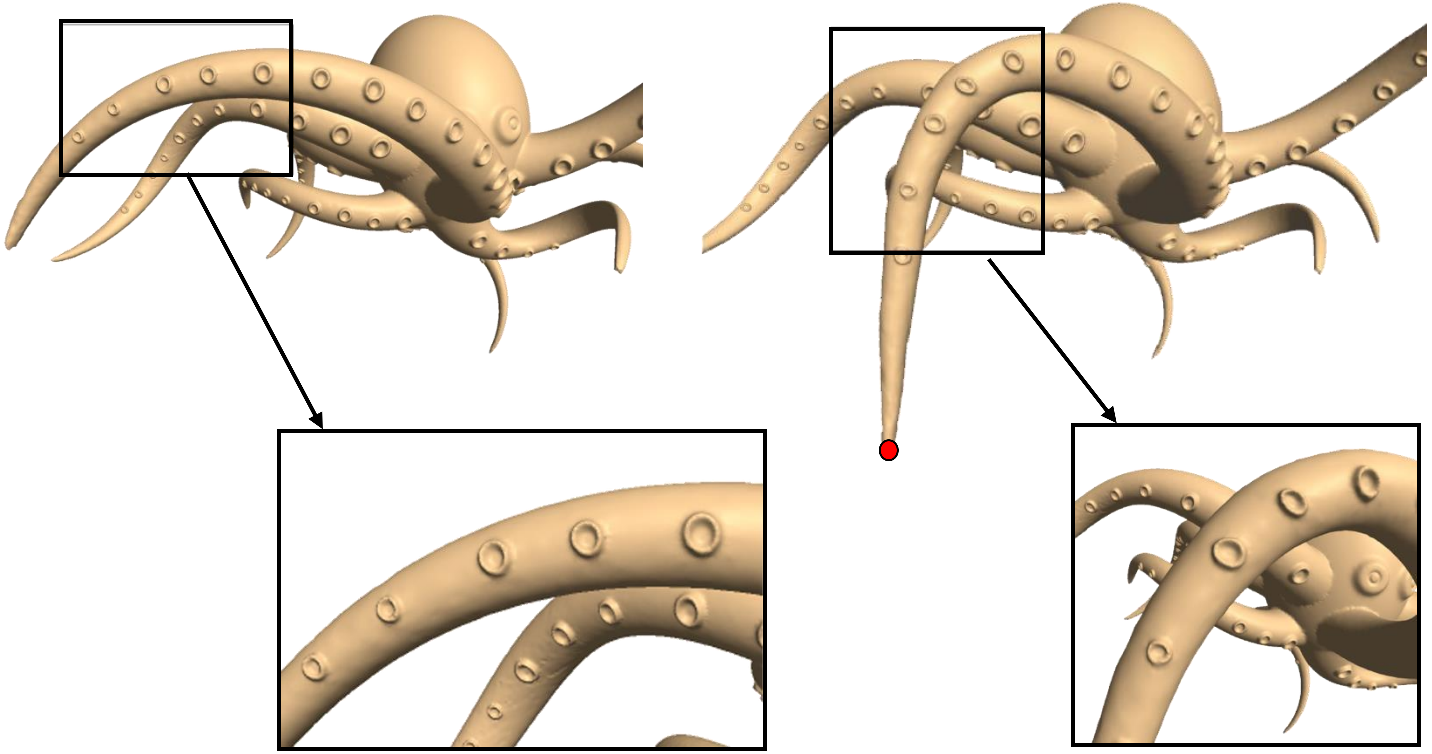
- Edit the surface using the original Laplacians δ (naïve Laplacian editing)
- Compute smoothed local frames, estimate rotation



Estimation of Rotations

Lipman et al. 2004

- Then solve the optimization again with the **rotated** δ's!



$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - \delta_i\|^2$$

$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - R_i\delta_i\|^2$$

Estimation of Rotations

[Lipman et al. 2004]

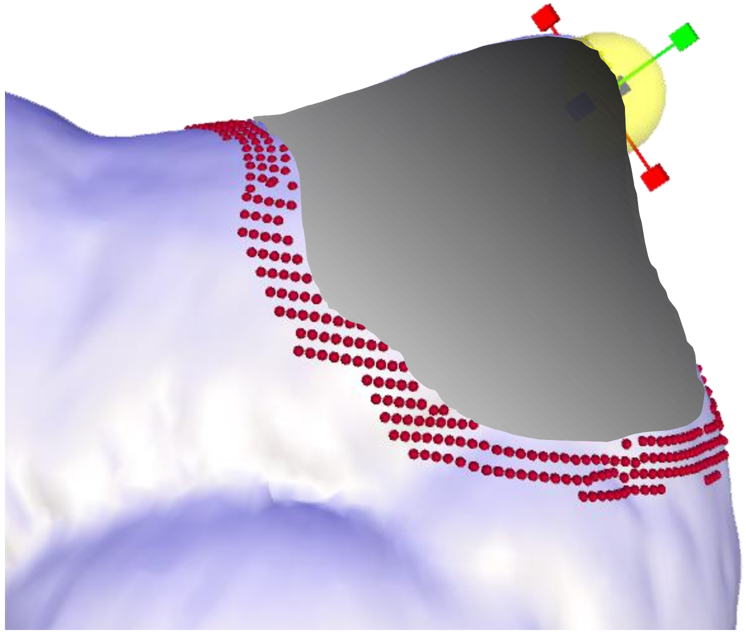
- Advantages:
  - Sparse linear solve
  - Less or no self-intersections thanks to global optimization (no more local displacements that fight each other)

- Disadvantages:
  - Heuristic estimation of the rotations
  - Speed depends on the support of the smooth local frame estimation operator; for highly detailed surfaces it must be large
  - Unclear how much we need to smooth (what is detail?)

Rotation Propagation

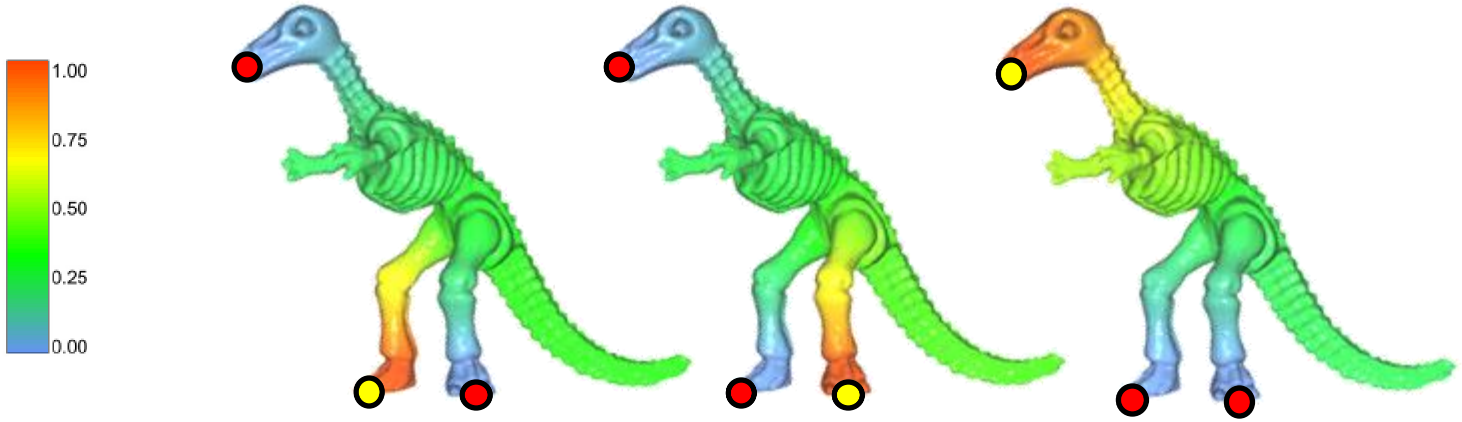
[Yu et al. SIGGRAPH 2004][Zayer et al. EG 2005][Lipman et al. SIGGRAPH 2005]

- Assume more user input: the user also specifies handle rotation, not just translation
- The rotation is diffused to the rest of the ROI



Rotation Propagation

- Geodesic distance [Yu et al. 2004]
- Harmonic field [Zayer et al. 2005]
- Optimization [Lipman et al. 2005, 2007]



Harmonic field

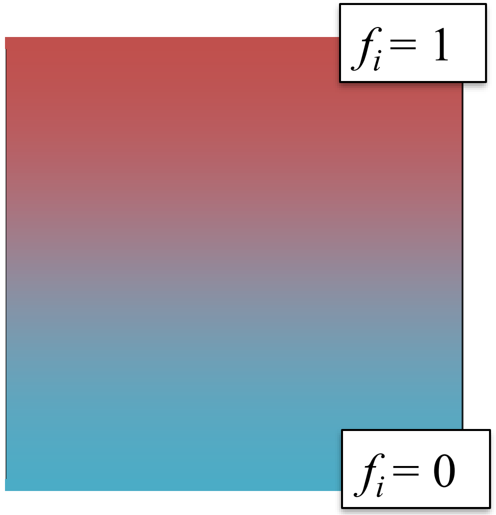
Harmonic Fields on Meshes

- Scalar function, attains 1 on one of the handles, 0 on all the other handles
- Smooth away from handles, no local extrema (maximum principle)
- Solve:

$$\Delta \mathbf{f} = 0$$

with constraints $f_i = 1$ on one handle,
$f_i = 0$ on all other handles

In this simple case,
the harmonic field
is a just a linear ramp



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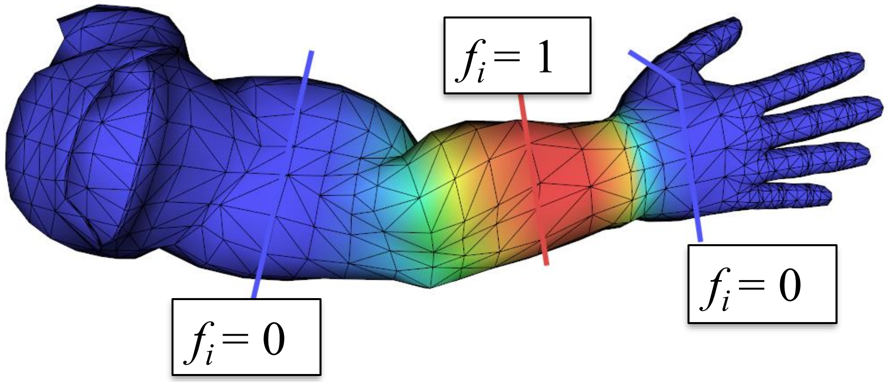
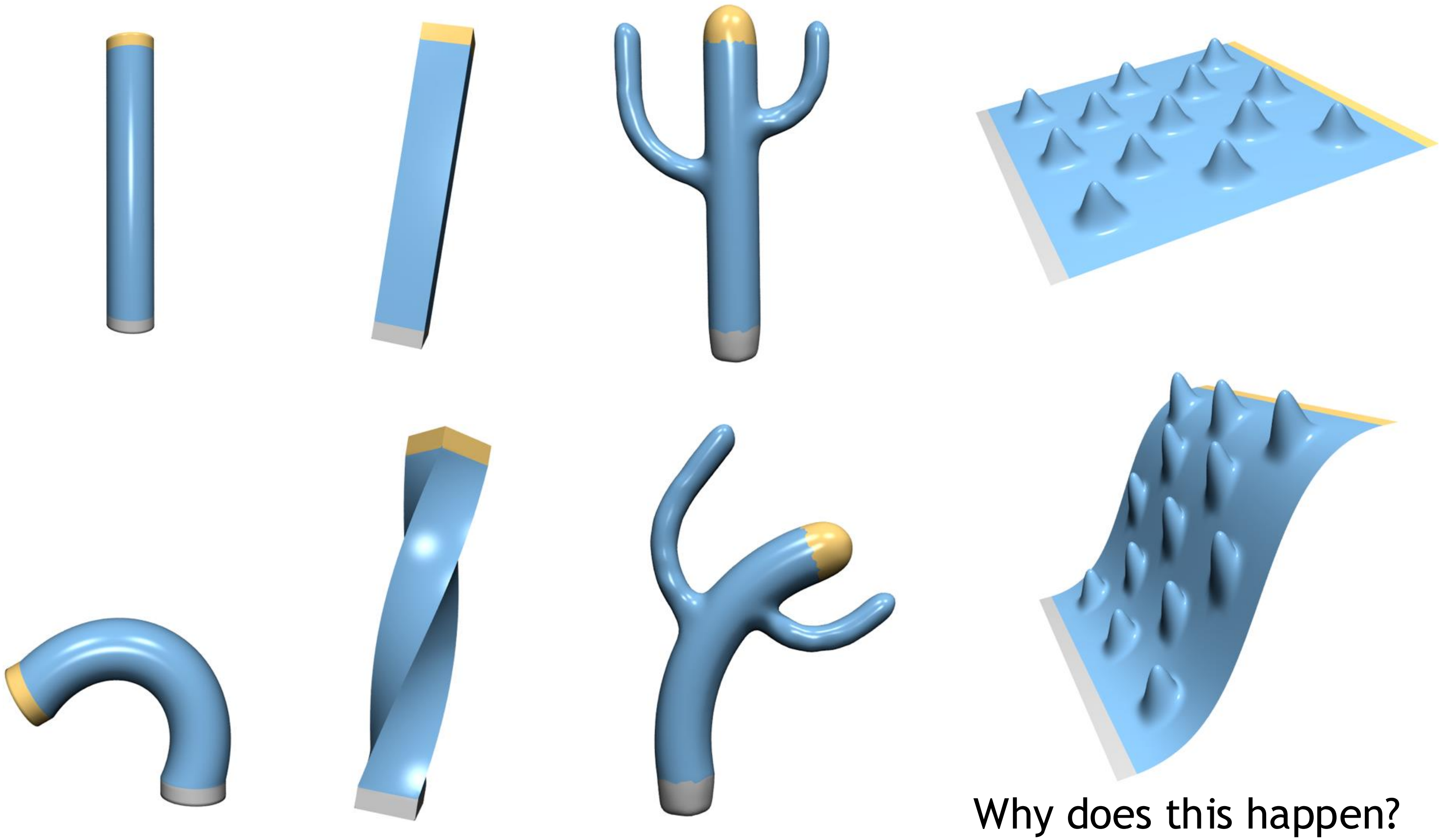
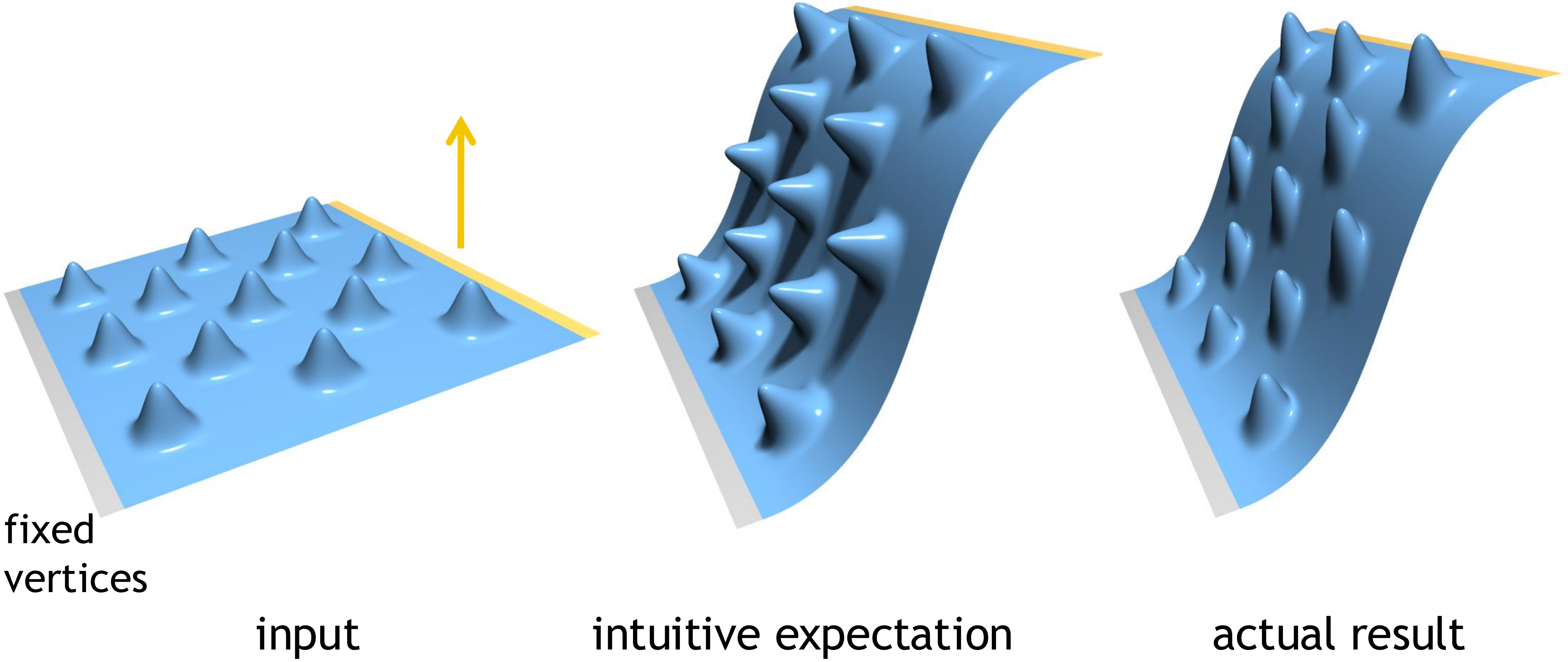


Rotation Propagation w/Harmonic Fields



Why does this happen?

Fundamental Problem: Invariance to Transformations



Rotation Propagation w/Harmonic Fields

- If rotations are provided and consistent with the desired transformation, this works well
- However, the method is translation-insensitive (doesn’t generate rotations when there are none provided)

Literature – rotation propagation

- Yu et al. 2004: **Mesh Editing with Poisson-Based Gradient Field Manipulation,** ACM SIGGRAPH 2004
- Lipman et al. 2005: **Linear Rotation-Invariant Coordinates for Meshes,** ACM SIGGRAPH 2005
- Zayer et al. 2005: **Harmonic Guidance for Surface Deformation,** EUROGRAPHICS 2005
- Lipman et al. 2007: **Volume and Shape Preservation via Moving Frame Manipulation,** ACM Transactions on Graphics 26(1), 2007

Thank You!

INTERACTIVE GEOMETRY LAB

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich