252-0538-00L, Spring 2025

Shape Modeling and Geometry Processing

Space Deformations

INTERACTIVE GEOMETRY LAB

May 14, 2025

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Space Deformation

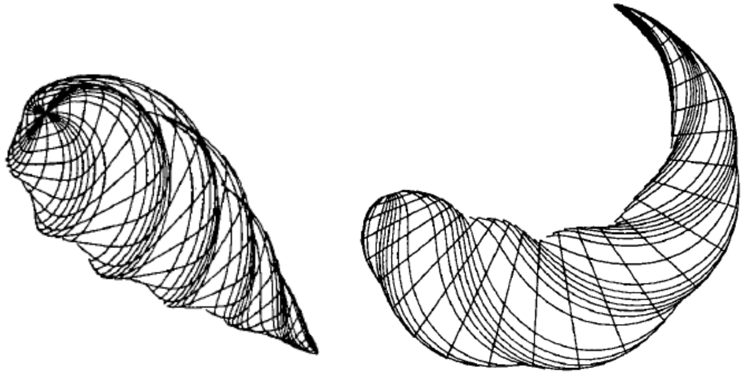
- Displacement function defined on the ambient space

$$\mathbf{d} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

- Evaluate the function on the points of the shape embedded in the space

$$\mathbf{x}' = \mathbf{x} + \mathbf{d}(\mathbf{x})$$

Twist warp
Global and local deformation of solids
[A. Barr, SIGGRAPH 84]



Space Deformation

- Displacement function defined on the ambient space

$$\mathbf{d} : \mathbb{R}^3 \to \mathbb{R}^3$$

- Evaluate the function on the points of the shape embedded in the space

$$\mathbf{x}' = \mathbf{x} + \mathbf{d}(\mathbf{x})$$

- Or simply define a warping of the entire space:

$$\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3; \quad \mathbf{x}' = \mathbf{f}(\mathbf{x})$$

Works with many shape reps!

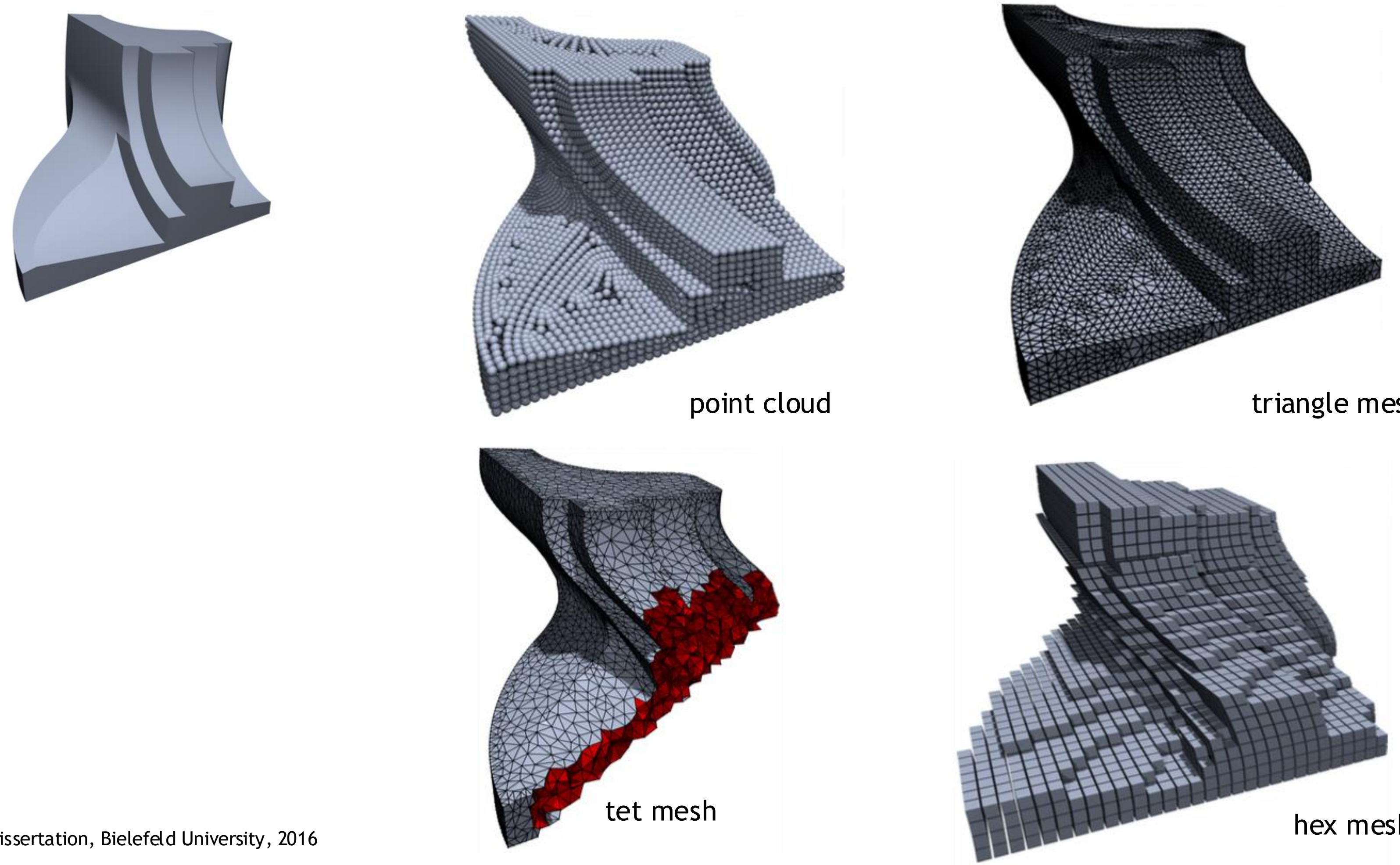


Image source: Daniel Sieger, PhD dissertation, Bielefeld University, 2016

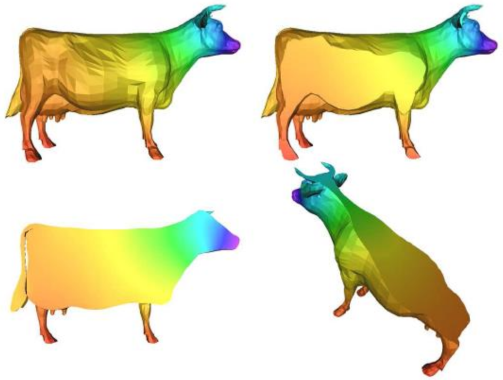
Freeform Deformations

- Control object enclosing the mesh

- User defines displacements $\mathbf{d}_i$ or new positions $\mathbf{p'}_i$ for each element of the control object

- Displacements (or new positions) are interpolated to the entire space using basis functions $B_i(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{d}(\mathbf{x}) = \sum_{i=1}^{k} \mathbf{d}_i \, B_i(\mathbf{x})$$

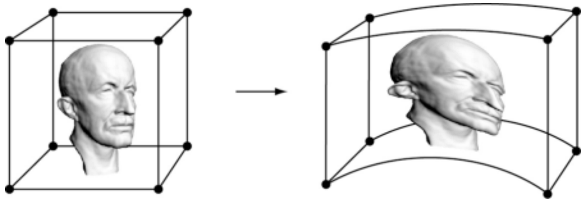


- Basis functions should be smooth for aesthetic results

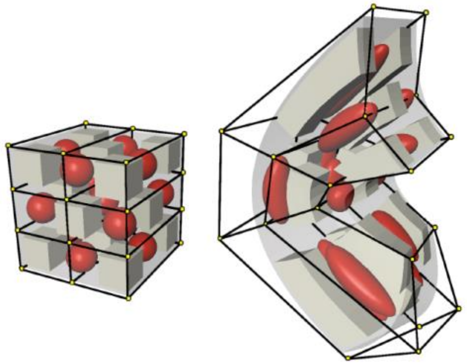
Freeform Deformation

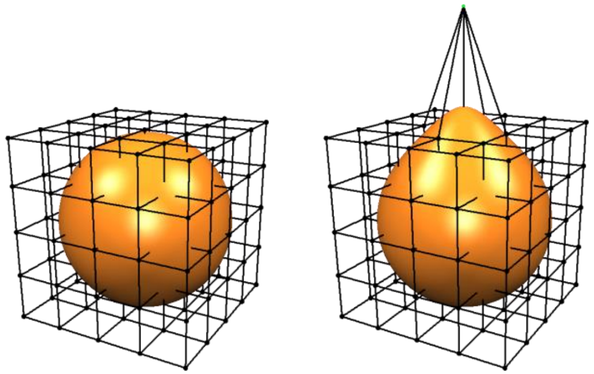
[Sederberg and Parry 86]

- Control object = lattice
- Basis functions $B_i(\mathbf{x})$ are trivariate tensor-product splines:



$$\mathbf{d}(x,y,z) = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} \mathbf{d}_{ijk}\, N_i(x) N_j(y) N_k(z)$$



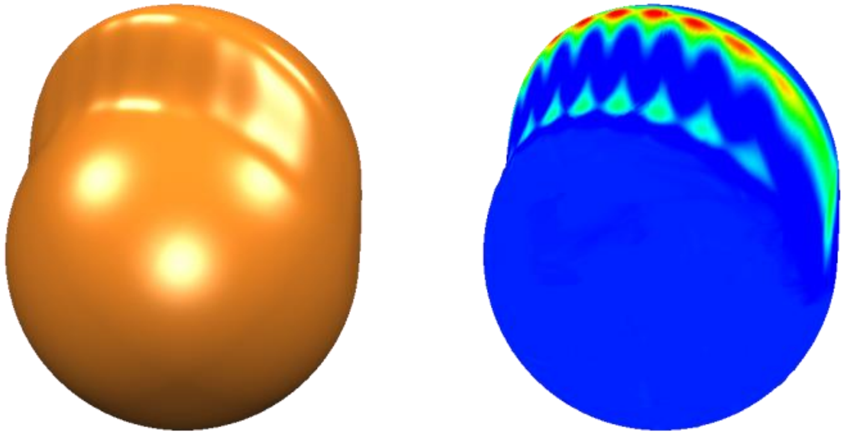


http://dl.acm.org/citation.cfm?id=15903

Freeform Deformation

[Sederberg and Parry 86]

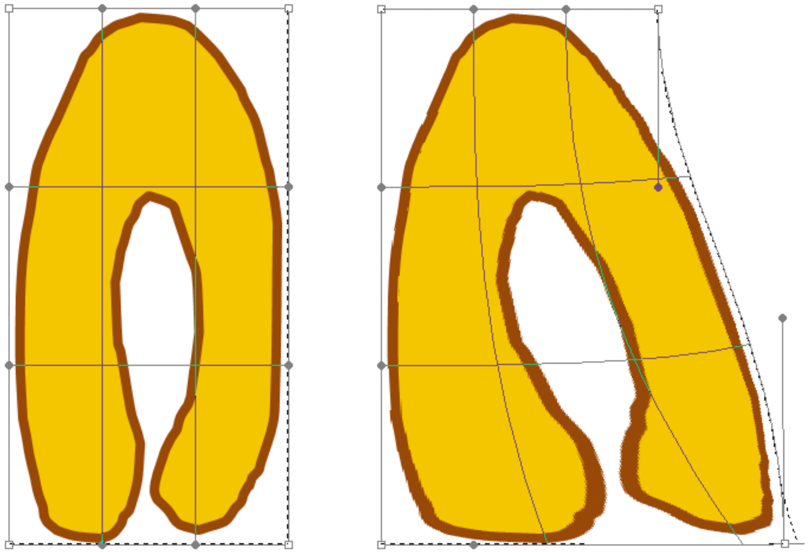
- Aliasing artifacts
- Interpolate deformation constraints on the surface?
  - Only in least squares sense



http://dl.acm.org/citation.cfm?id=15903

Limitations of Lattices as Control Objects

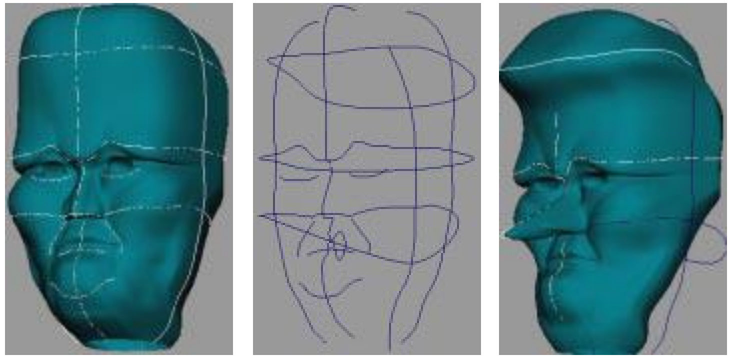
- Difficult to manipulate
- The control object is not related to the shape of the edited object
- Parts of the shape in close Euclidean distance always deform similarly, even if geodesically far

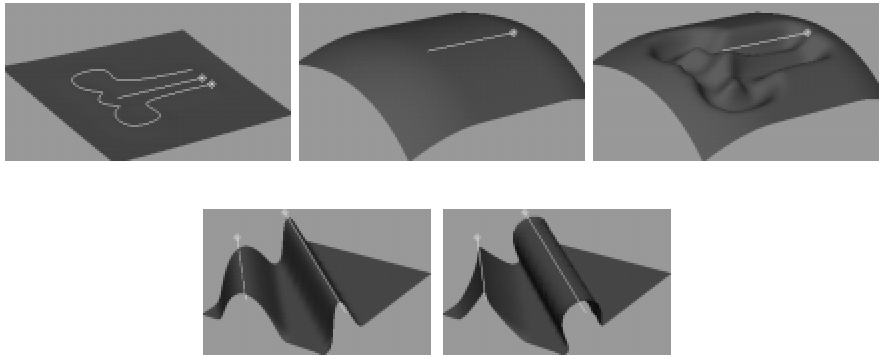


Wires

[Singh and Fiume 98]

- Control objects are arbitrary space curves
- Can place curves along meaningful features of the edited object
- Smooth deformations around the curve with decreasing influence





http://www.dgp.toronto.edu/~karan/pdf/ksinghpaperwire.pdf

Handle Metaphor

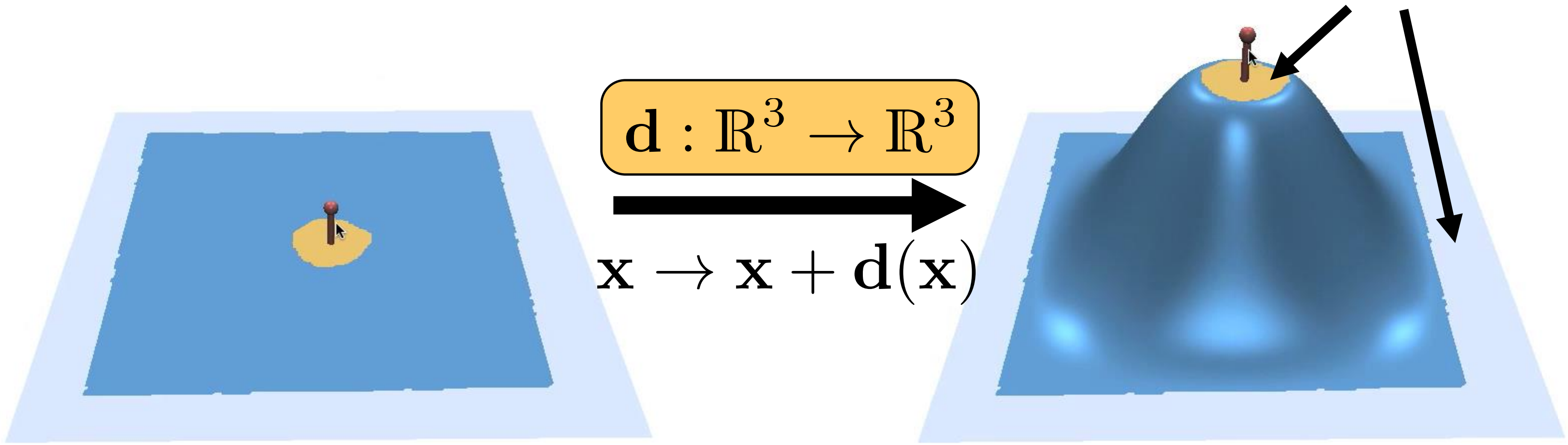
[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

- Wish list for the displacement function $\mathbf{d}(\mathbf{x})$:
  - Interpolate prescribed constraints
  - Smooth, intuitive deformation

$$\mathbf{d}(\mathbf{x}_i) = \mathbf{d}_i$$

$$\mathbf{d} : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\mathbf{x} \to \mathbf{x} + \mathbf{d}(\mathbf{x})$$



Volumetric Energy Minimization

[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

- Minimize similar energies to surface case

$$\int_{\mathbb{R}^3} \|\mathbf{d}_{xx}\|^2 + \|\mathbf{d}_{xy}\|^2 + \ldots + \|\mathbf{d}_{zz}\|^2 \; dx\,dy\,dz \rightarrow \min$$

- But displacement function lives in 3D...
  - Need a volumetric space tessellation?
  - No, same functionality provided by RBFs!

Radial Basis Functions

[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

- Represent deformation by RBFs

$$\mathbf{d}(\mathbf{x}) = \sum_j \mathbf{w}_j \, \varphi(\|\mathbf{c}_j - \mathbf{x}\|) + \mathbf{P}(\mathbf{x})$$

- Triharmonic basis function $\varphi(r) = r^3$
  - C^2 boundary constraints
  - Highly smooth / fair interpolation

$$\int_{\mathbb{R}^3} \|\mathbf{d}_{xxx}\|^2 + \|\mathbf{d}_{xyy}\|^2 + \ldots + \|\mathbf{d}_{zzz}\|^2 \; dx\,dy\,dz \to \min$$

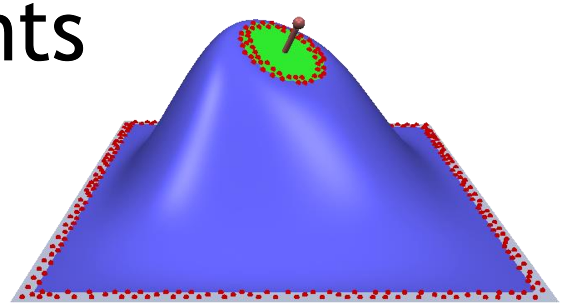
Radial Basis Functions

[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

- Represent deformation by RBFs

$$\mathbf{d}(\mathbf{x}) = \sum_j \mathbf{w}_j \, \varphi(\|\mathbf{c}_j - \mathbf{x}\|) + \mathbf{P}(\mathbf{x})$$

- RBF fitting
  - Interpolate displacement constraints
  - Solve linear system for $\mathbf{w}_j$ and $\mathbf{P}$ ($\mathbf{P}$ is a linear polynomial)



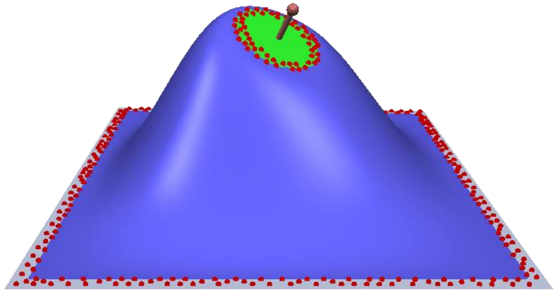
Radial Basis Functions

[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]

- Represent deformation by RBFs

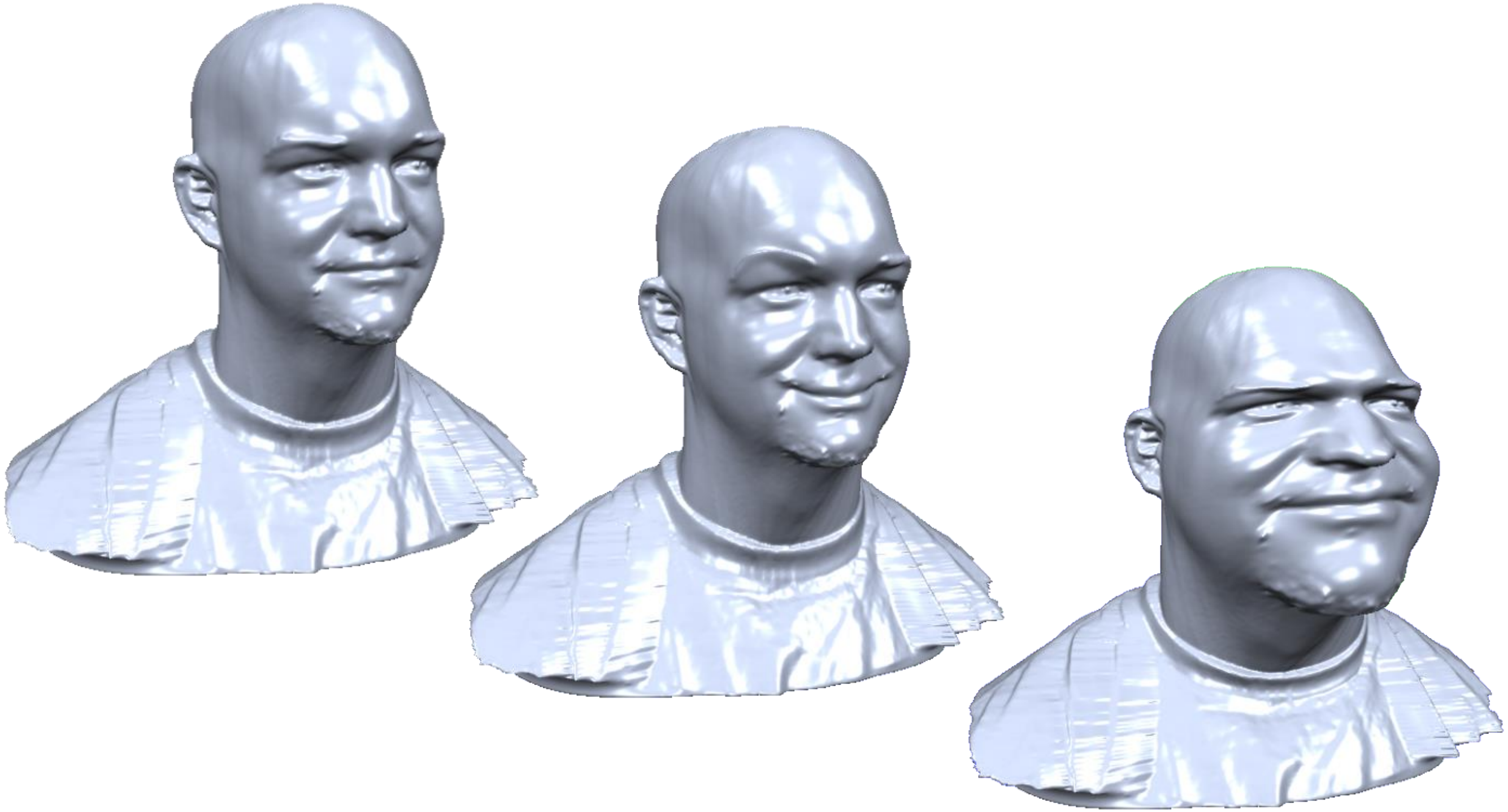
$$\mathbf{d}(\mathbf{x}) = \sum_j \mathbf{w}_j \, \varphi(\|\mathbf{c}_j - \mathbf{x}\|) + \mathbf{P}(\mathbf{x})$$

- RBF evaluation
  - Function $\mathbf{d}$ transforms points
  - Jacobian^{-T} $\nabla\mathbf{d}^{-T}$ transforms normals
  - Precompute basis functions
  - Evaluate on the GPU!



Local & Global Deformations

[Real-Time Shape Editing using Radial Basis Functions, Botsch and Kobbelt, EUROGRAPHICS 2005]



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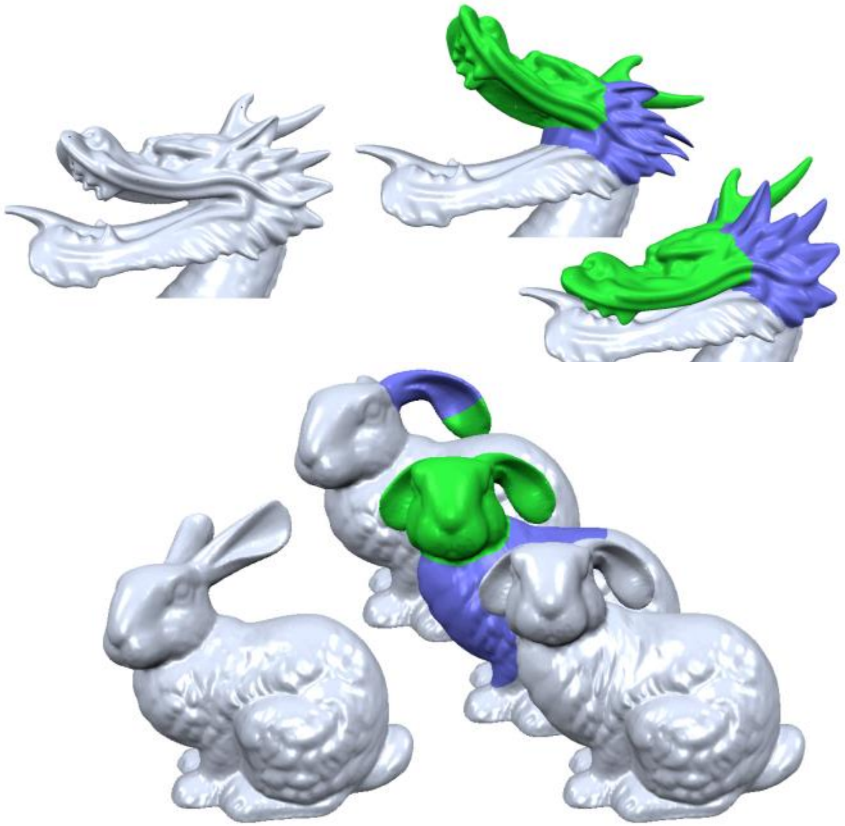
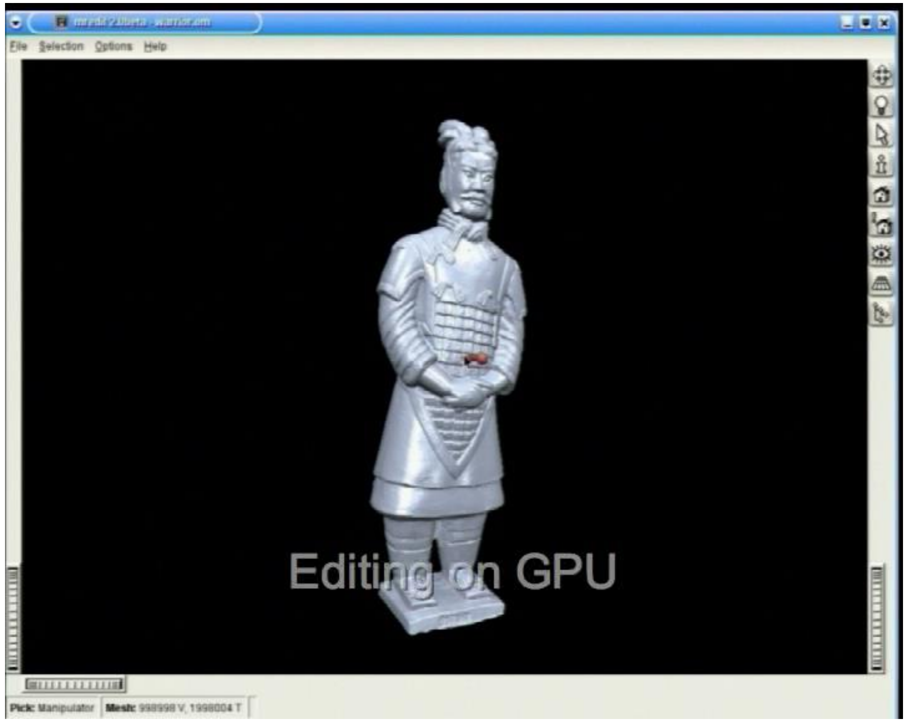


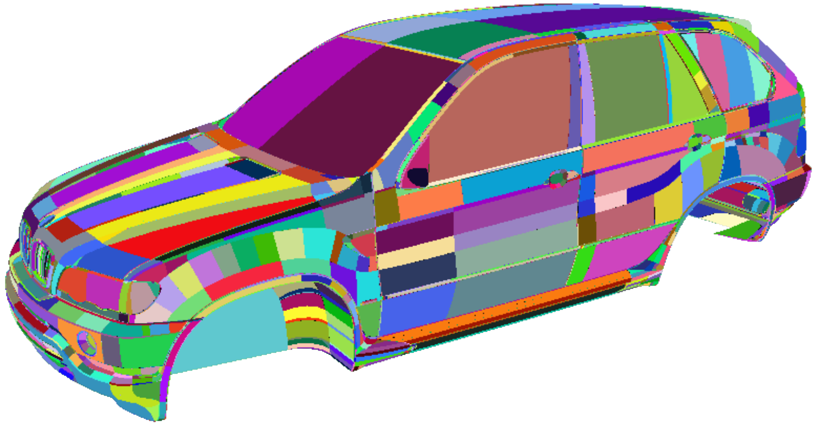


1M vertices
movie

Space Deformations

Summary so far

- Handle arbitrary input
  - Meshes (also non-manifold)
  - Point sets
  - Polygonal soups
  - …
- Complexity mainly depends on the control object, not the surface



- 3M triangles
- 10k components
- Not oriented
- Not manifold

Space Deformations

Summary so far

- Handle arbitrary input
  - Meshes (also non-manifold)
  - Point sets
  - Polygonal soups
  - …

- Easier to analyze:
  functions on Euclidean domain
  - Volume preservation: |Jacobian| = 1

$$\mathbf{F}(x,y,z) = (F(x,y,z), G(x,y,z), H(x,y,z))$$

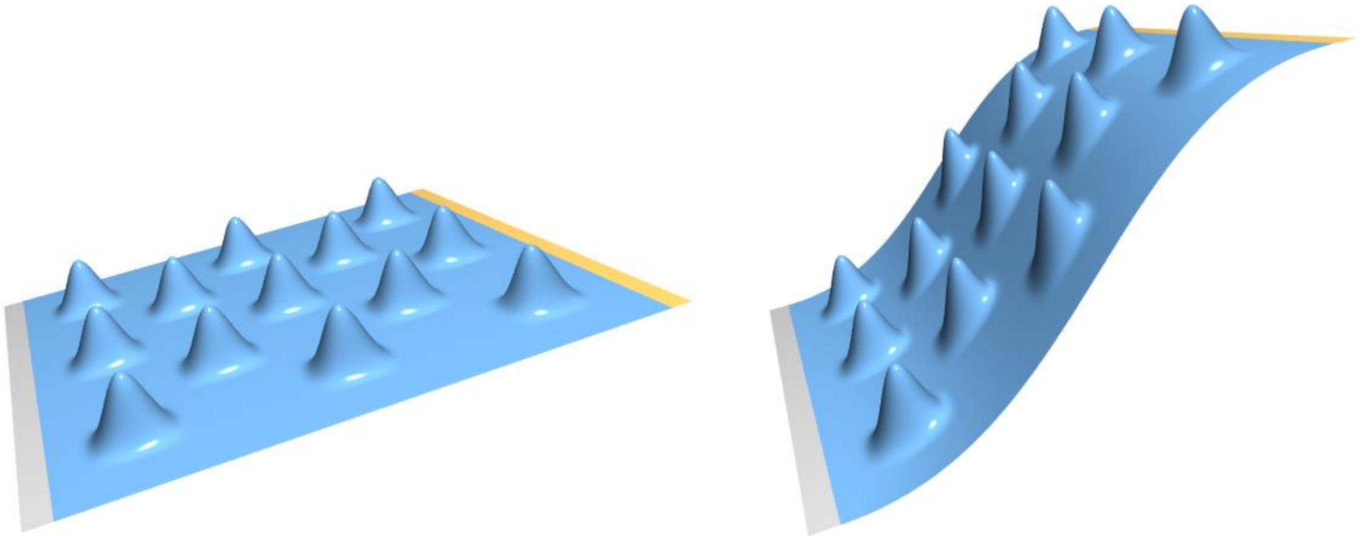
then the Jacobian is the determinant

$$Jac(\mathbf{F}) = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{vmatrix}$$

Space Deformations

Summary so far

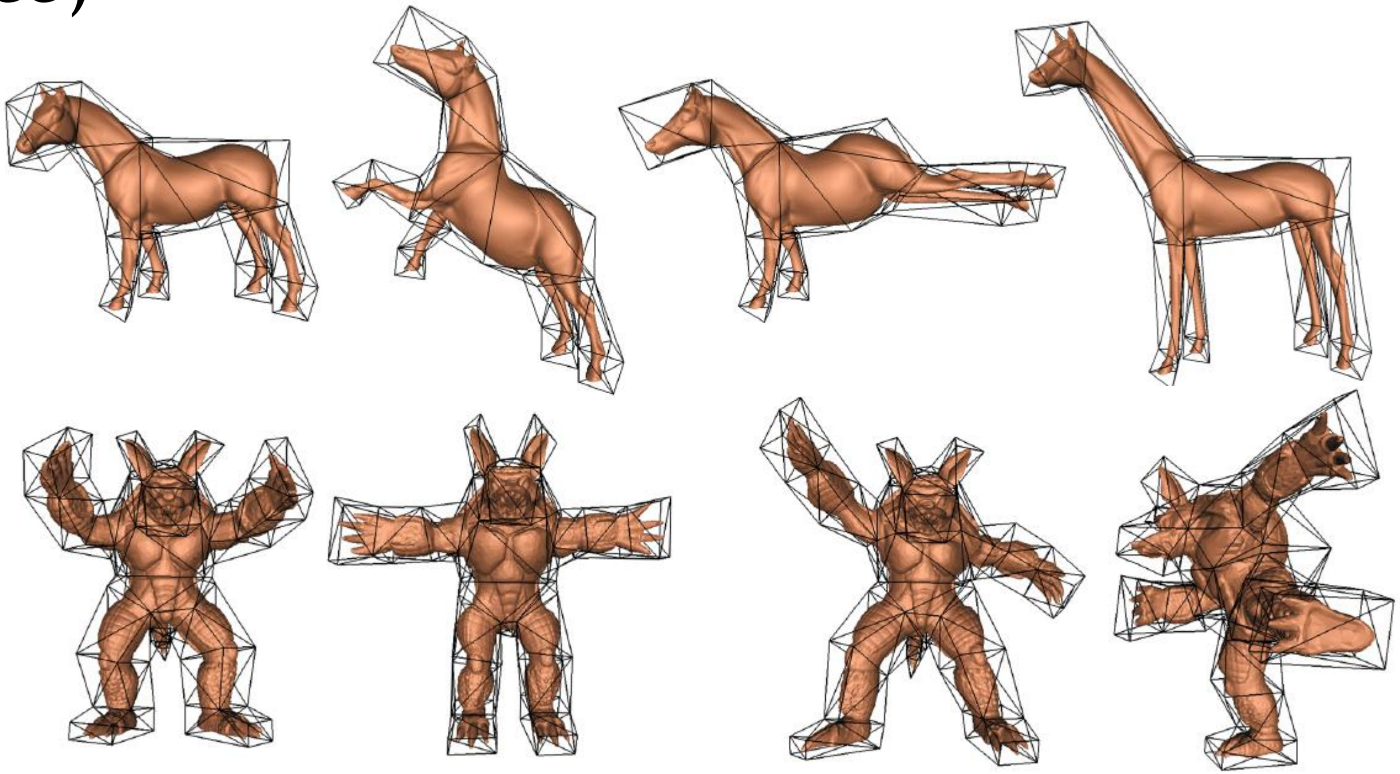
- The deformation is only loosely aware of the shape that is being edited
- Small Euclidean distance → similar deformation
- Local surface detail may be distorted



Cage-based Deformations

[Ju et al. 2005]

- Cage = crude version of the input shape
- Polyhedron (not a lattice)

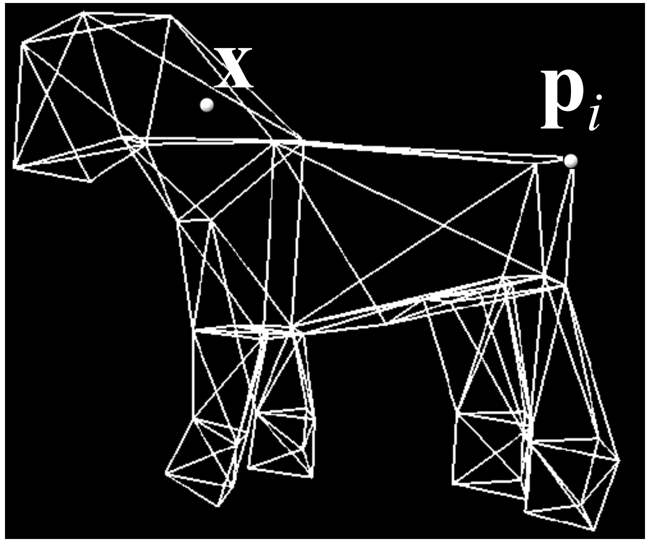


http://www.cs.wustl.edu/~taoju/research/meanvalue.pdf

Cage-based Deformations

[Ju et al. 2005]

- Each point $\mathbf{x}$ in space is represented w.r.t. the cage elements using coordinate functions



$$\mathbf{x} = \sum_{i=1}^{k} w_i(\mathbf{x})\,\mathbf{p}_i$$

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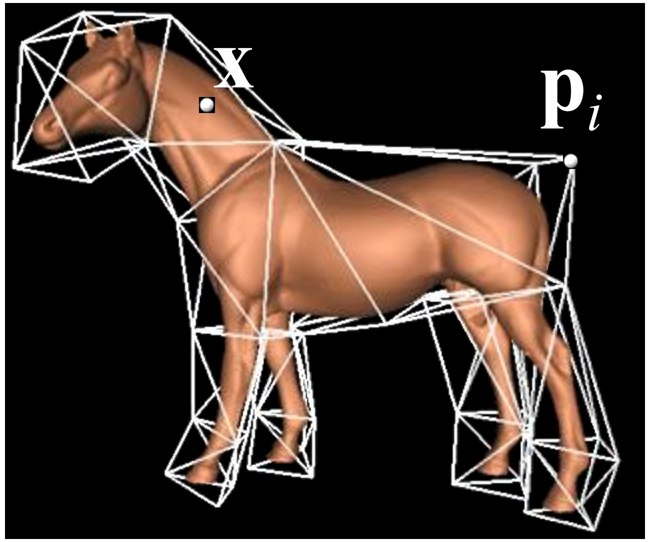

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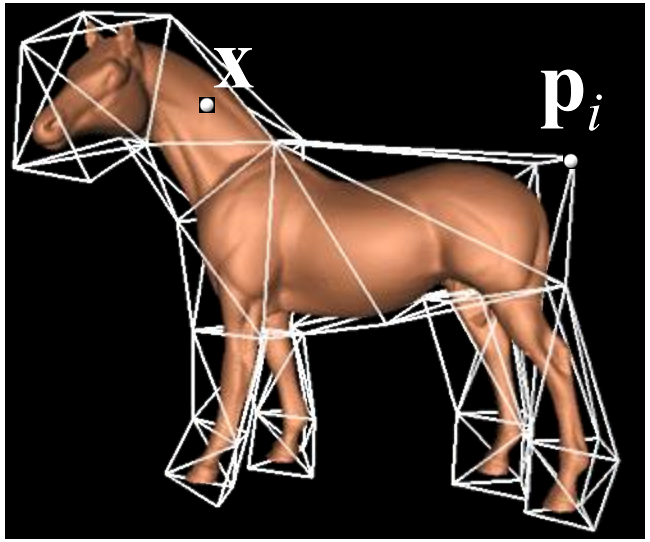

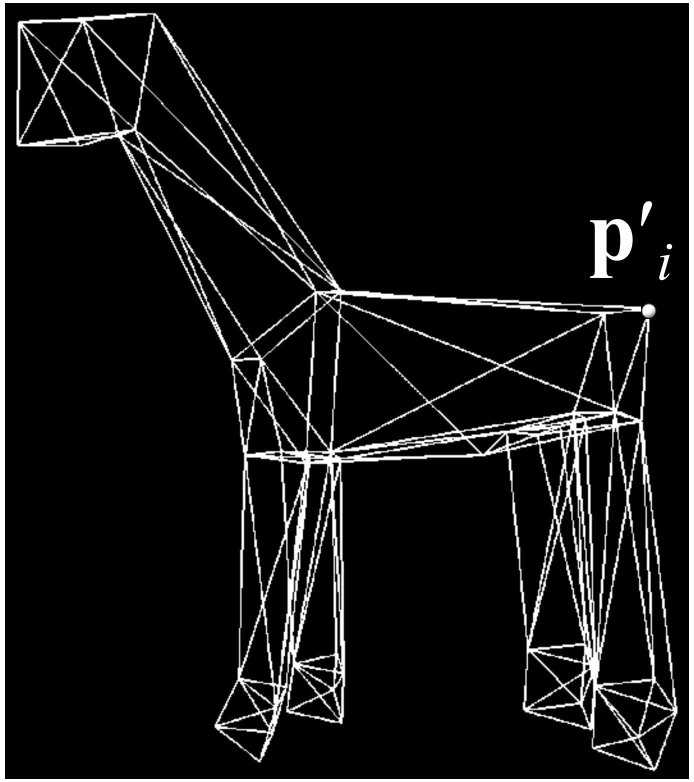

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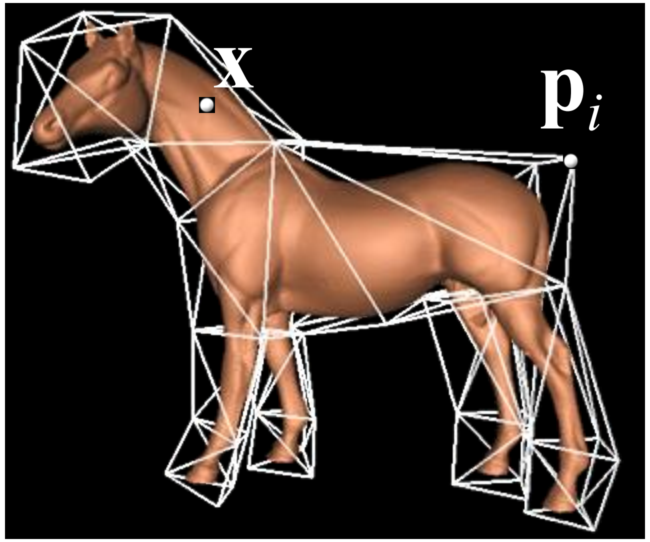
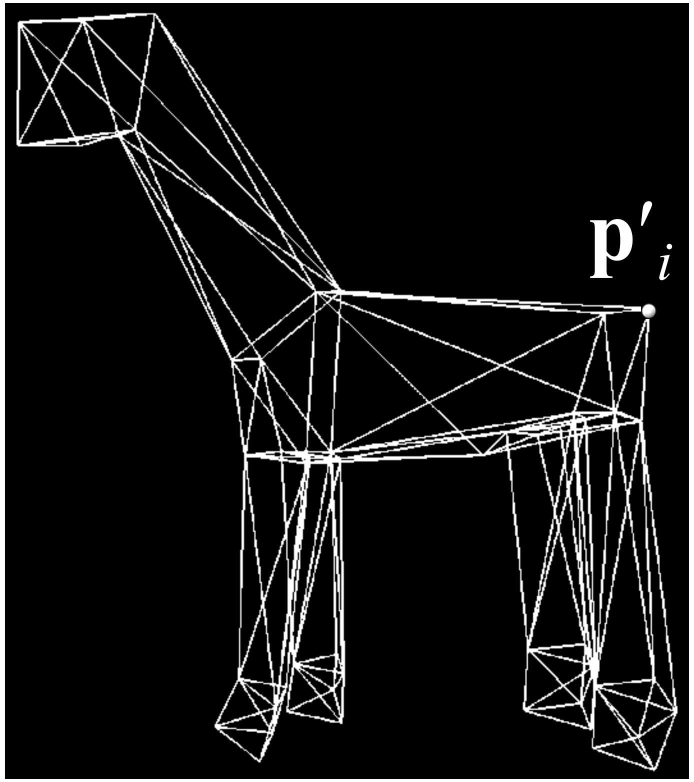
Cage-based Deformations

[Ju et al. 2005]

$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\,\mathbf{p}'_i$$





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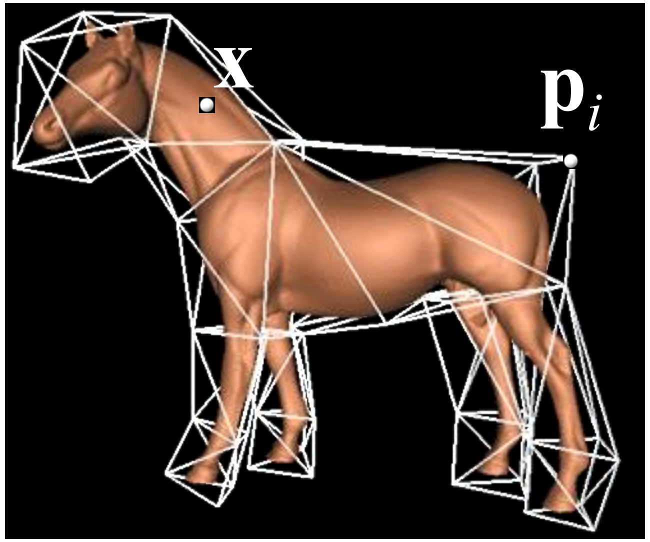
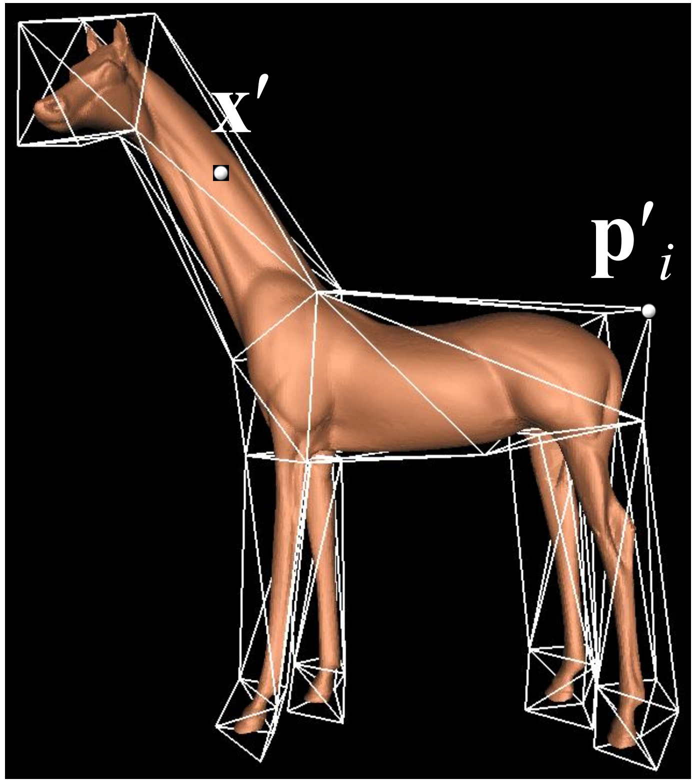




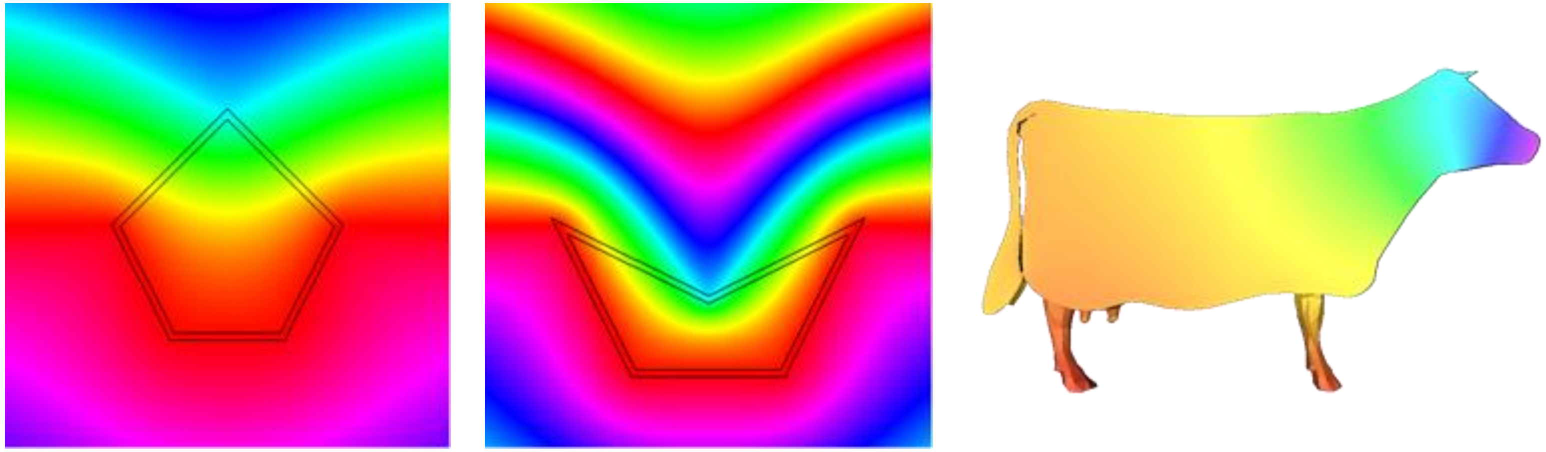
http://www.cs.wustl.edu/~taoju/research/meanvalue.pdf

Generalized barycentric coordinates

- Lagrange property: $w_i(\mathbf{p}_j) = \delta_{ij}$
- Reproduction: $\forall \mathbf{x},\ \sum_{i=1}^{k} w_i(\mathbf{x})\,\mathbf{p}_i = \mathbf{x}$
- Partition of unity: $\forall \mathbf{x},\ \sum_{i=1}^{k} w_i(\mathbf{x}) = 1$

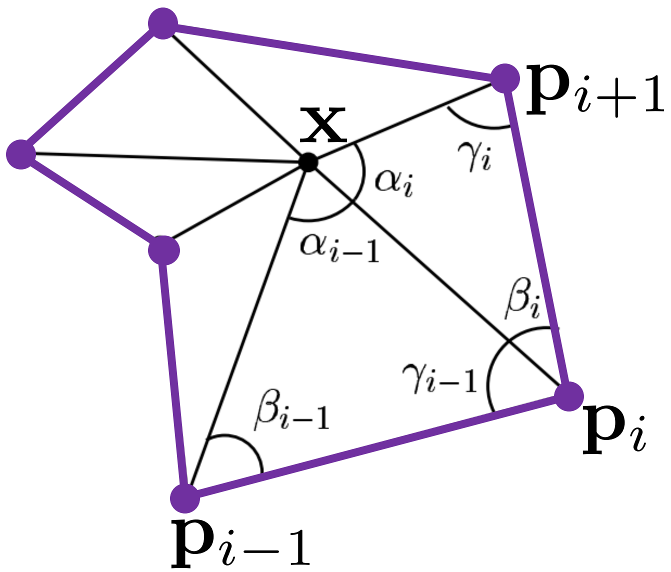
Coordinate Functions

- Mean-value coordinates
  [Floater 2003*, Ju et al. 2005]
  - Generalization of barycentric coordinates
  - Closed-form solution for $w_i(\mathbf{x})$



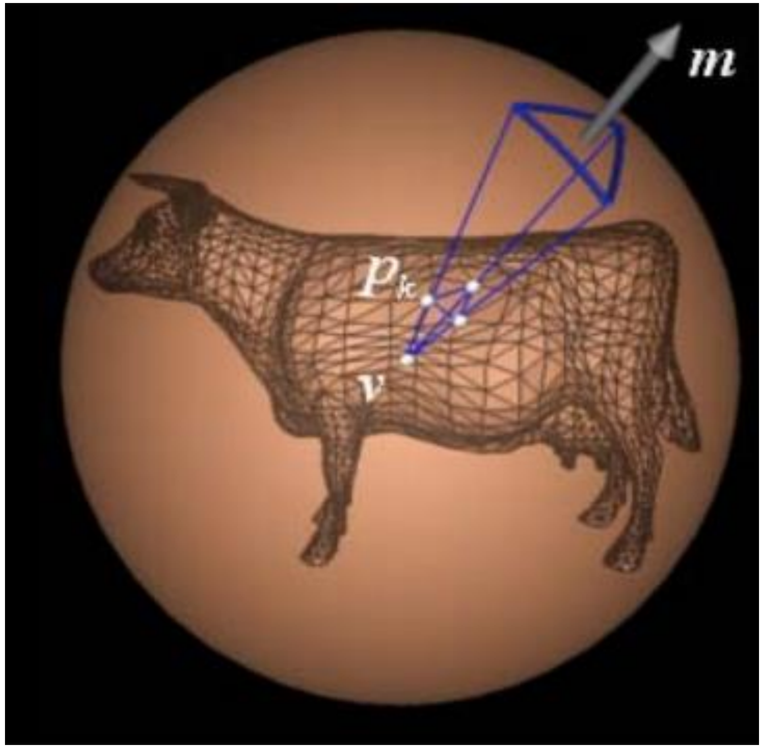
* Michael Floater, “Mean value coordinates”, CAGD 20(1), 2003

2D Mean Value Coordinates



$$\lambda_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{\|\mathbf{p}_i - \mathbf{x}\|}; \quad w_i(\mathbf{x}) = \frac{\lambda_i(\mathbf{x})}{\sum_{j=1}^{k} \lambda_j(\mathbf{x})}$$

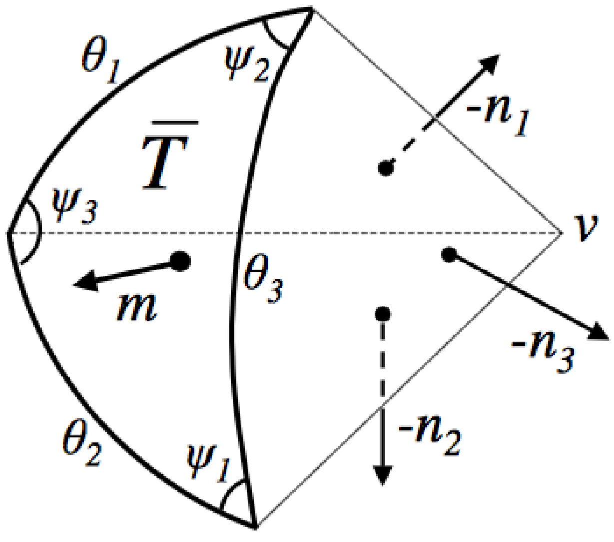
3D Mean Value Coordinates



Mean Value Coordinates for Closed Triangular Meshes
Tao Ju, Scott Schaefer, Joe Warren

// Robust evaluation on a triangular mesh
for each vertex p_j with values f_j
    $d_j \leftarrow \|p_j - x\|$
    if $d_j < \varepsilon$ return f_j
    $u_j \leftarrow (p_j - x)/d_j$
$totalF \leftarrow 0$
$totalW \leftarrow 0$
for each triangle with vertices p_1, p_2, p_3 and values f_1, f_2, f_3
    $l_i \leftarrow \|u_{i+1} - u_{i-1}\|$ *// for* $i = 1,2,3$
    $\theta_i \leftarrow 2\arcsin[l_i/2]$
    $h \leftarrow (\sum \theta_i)/2$
    if $\pi - h < \varepsilon$
        // x lies on t, use 2D barycentric coordinates
        $w_i \leftarrow \sin[\theta_i] d_{i-1} d_{i+1}$
        return $(\sum w_i f_i)/(\sum w_i)$
    $c_i \leftarrow (2\sin[h]\sin[h-\theta_i])/(\sin[\theta_{i+1}]\sin[\theta_{i-1}]) - 1$
    $s_i \leftarrow \mathrm{sign}[det[u_1, u_2, u_3]]\sqrt{1 - c_i^2}$
    if $\exists i, |s_i| \leq \varepsilon$
        // x lies outside t on the same plane, ignore t
        continue
    $w_i \leftarrow (\theta_i - c_{i+1}\theta_{i-1} - c_{i-1}\theta_{i+1})/(d_i \sin[\theta_{i+1}] s_{i-1})$
    $totalF += \sum w_i f_i$
    $totalW += \sum w_i$
$f_x \leftarrow totalF/totalW$

3D Mean Value Coordinates



(b)

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    continue
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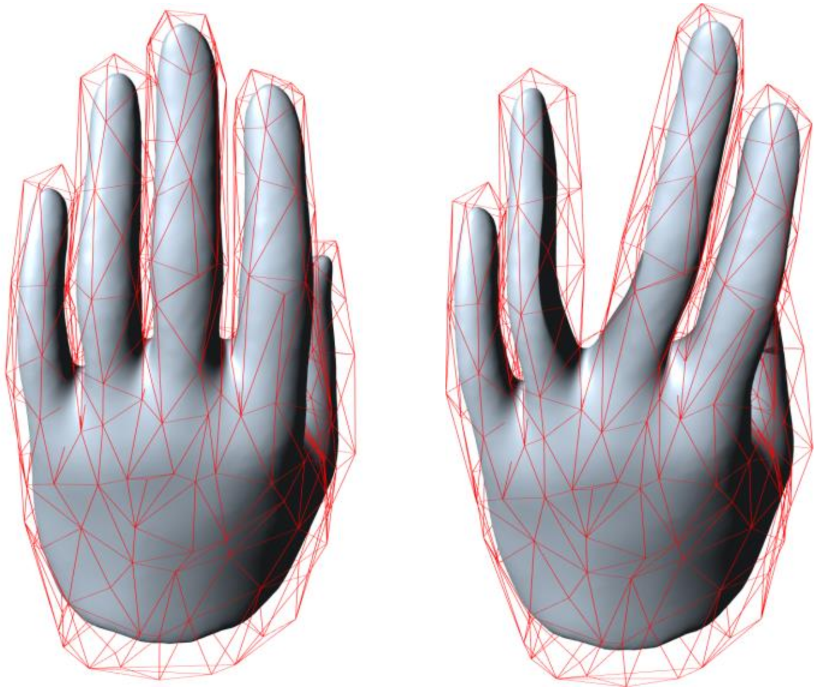
  $totalF += \sum w_i f_i$
  $totalW += \sum w_i$
$f_x \leftarrow totalF/totalW$

Mean Value Coordinates for Closed Triangular Meshes
Tao Ju, Scott Schaefer, Joe Warren

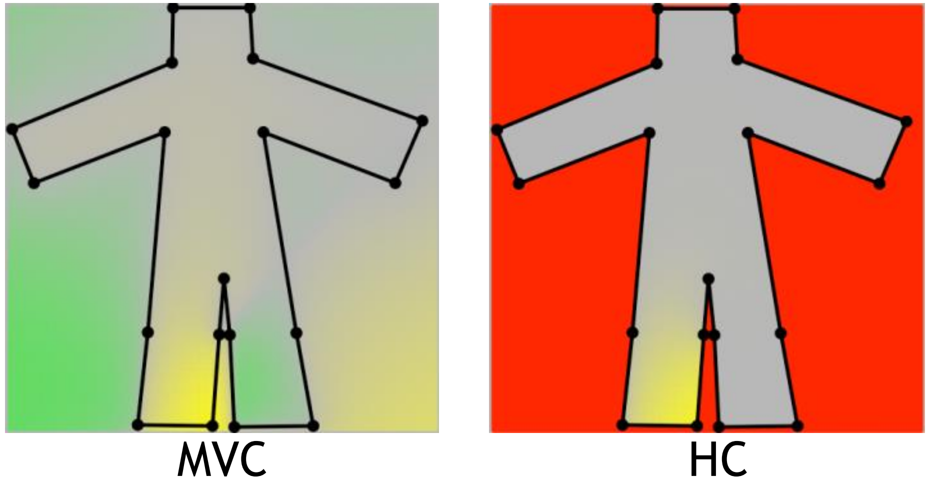
Coordinate Functions

- Mean-value coordinates
  [Floater 2003, Ju et al. 2005]
  - Not necessarily positive on non-convex domains



Coordinate Functions

- Harmonic coordinates ([Joshi et al. 2007](http://www.cs.jhu.edu/~misha/Fall07/Papers/Joshi07.pdf))
  - Harmonic functions $h_i(\mathbf{x})$ for each cage vertex $\mathbf{p}_i$
  - Solve $\Delta h_i = 0$
  - subject to: h_i linear on the boundary s.t. $h_i(\mathbf{p}_j) = \delta_{ij}$



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Coordinate Functions

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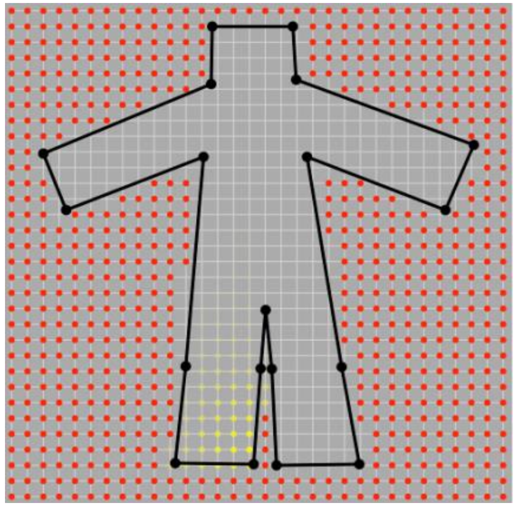
  - Solve $\Delta h_i = 0$
  - subject to: h_i linear on the boundary s.t. $h_i(\mathbf{p}_j) = \delta_{ij}$
- Volumetric Laplace equation
- Discretization, no closed-form



Coordinate Functions

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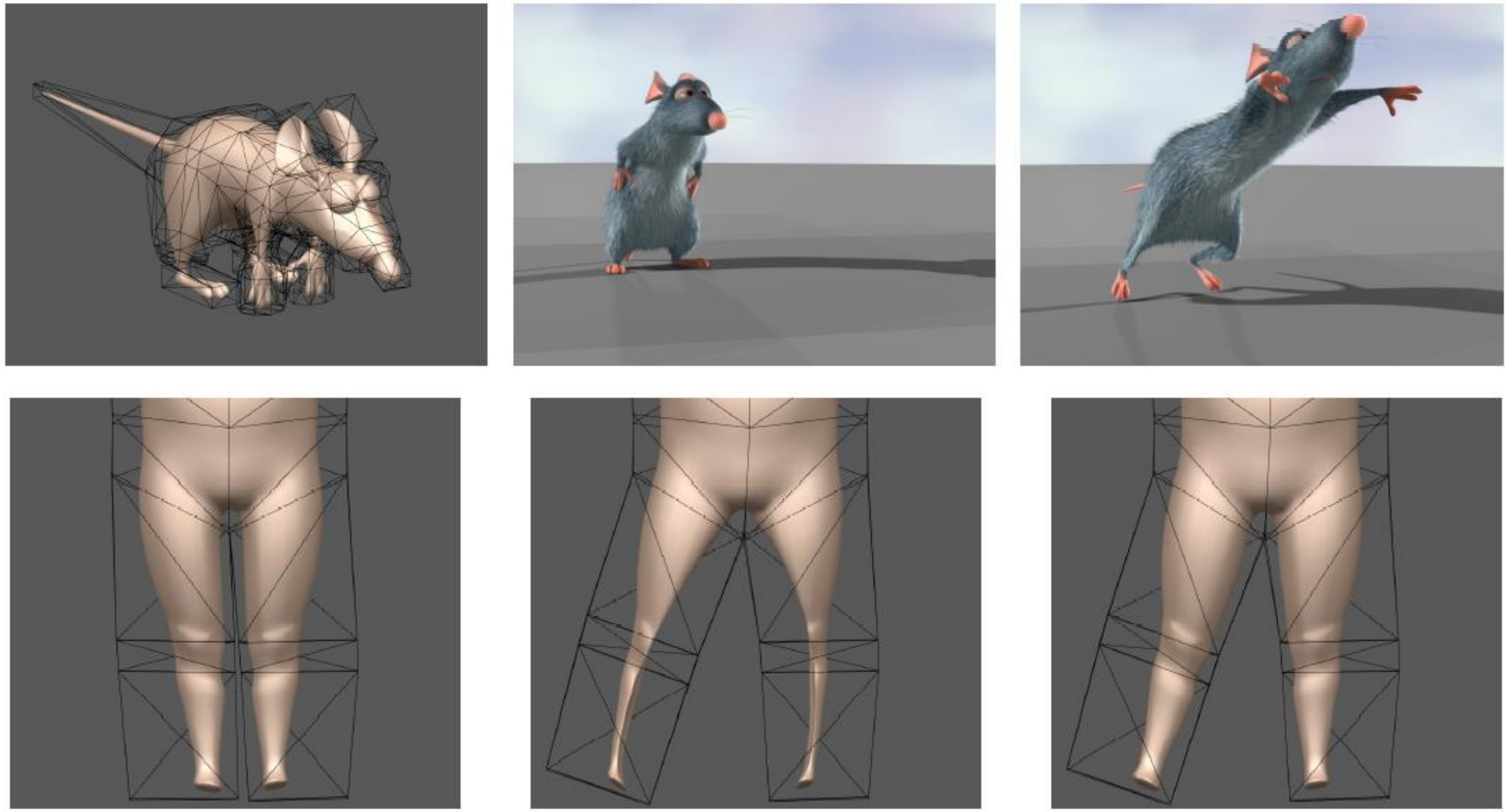
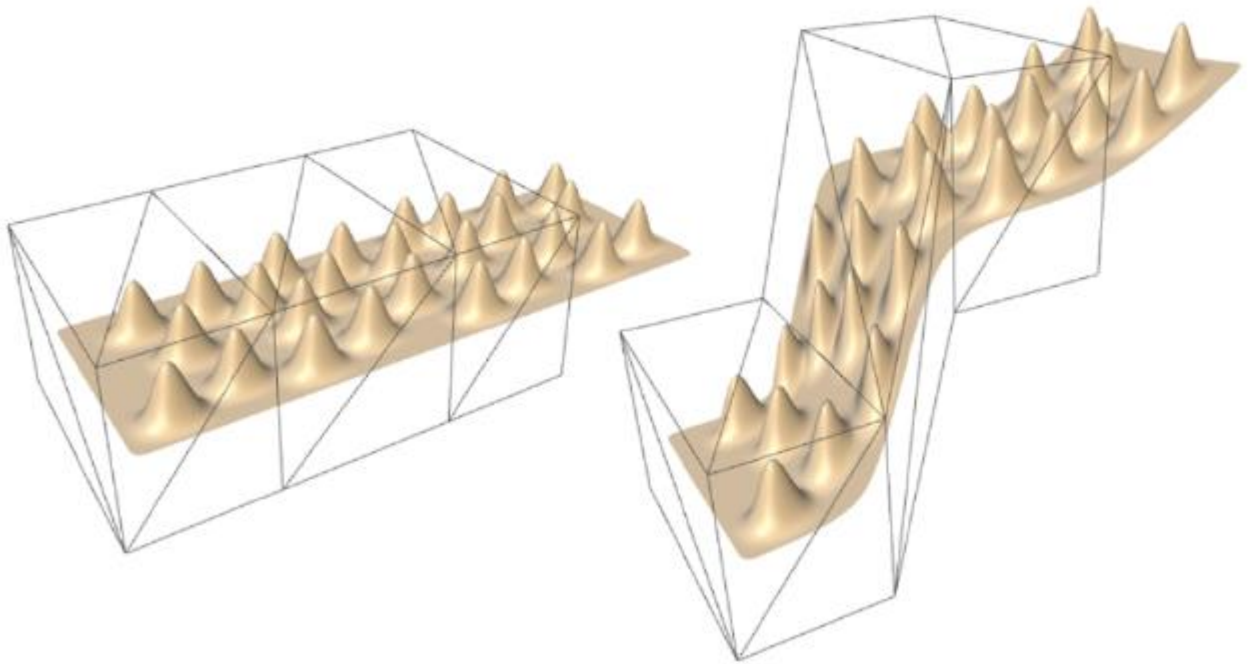


MVC HC

Coordinate Functions

- **Green coordinates** (Lipman et al. 2008)
- Observation: previous vertex-based basis functions always lead to affine-invariance!

$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\, \mathbf{p}'_i$$

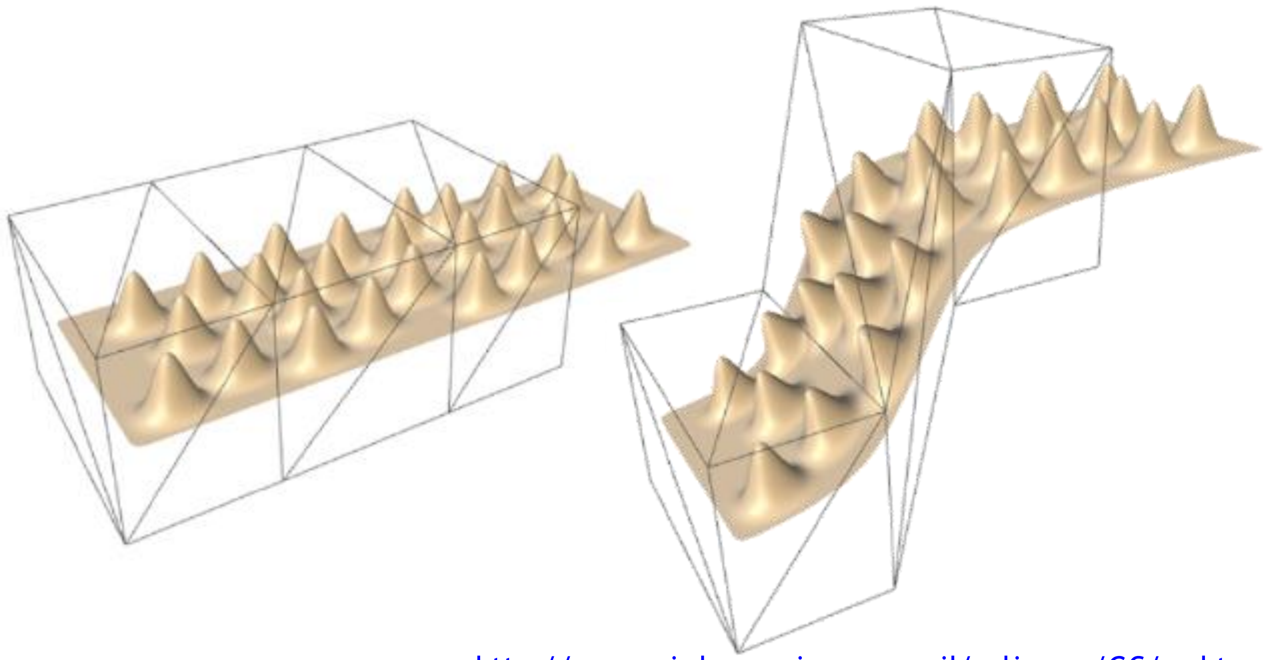


http://www.wisdom.weizmann.ac.il/~ylipman/GC/gc.htm

Coordinate Functions

- **Green coordinates** (Lipman et al. 2008)
- Correction: Make the coordinates depend on the cage faces as well

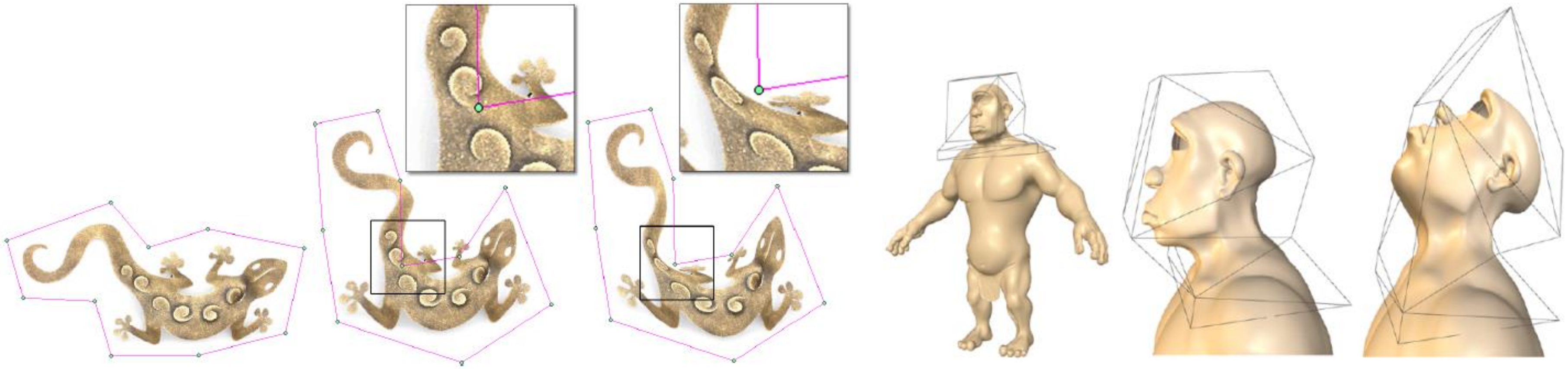
$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\,\mathbf{p}'_i + \sum_{j=1}^{m} \psi_j(\mathbf{x})\,\mathbf{n}'_j$$



http://www.wisdom.weizmann.ac.il/~ylipman/GC/gc.htm

Coordinate Functions

- **Green coordinates** (Lipman et al. 2008)
- Closed-form solution
- Conformal in 2D, quasi-conformal in 3D



GC MVC GC

Coordinate Functions

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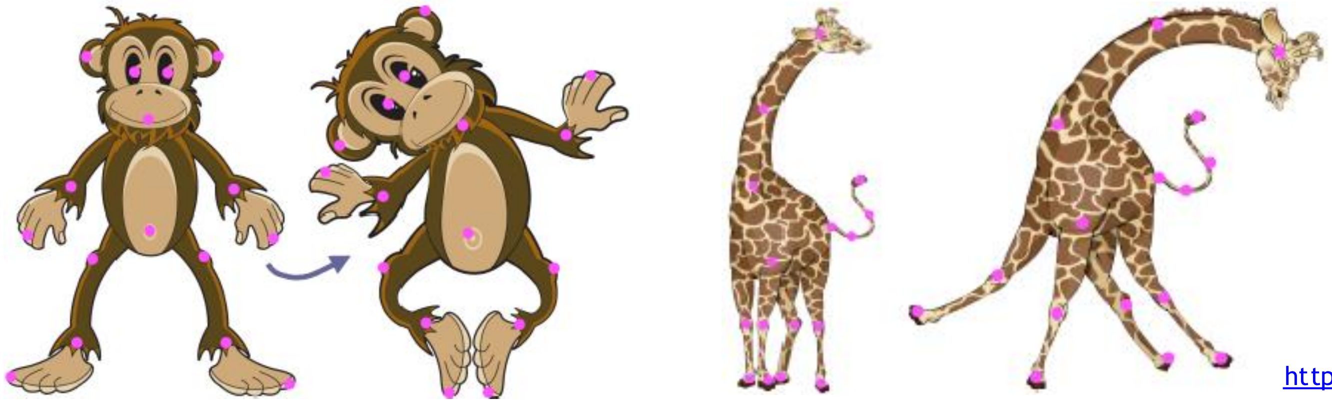
Alternative interpretation in 2D via holomorphic functions and extension to point handles : [Weber et al. Eurographics 2009]



http://www.eng.biu.ac.il/weberof/publications/complex-coordinates/

Cage-based methods: Summary

Pros:

- Nice control over volume
  - Squish/stretch
- Efficient GPU implementation

Cons:

- Hard to control local details of embedded surface
- Manual design of cage mesh

Neural cages

- Wang Yifan et al. CVPR 2020: use MVC and let a neural network find the cage vertex positions

Neural cages

- Wang Yifan et al. CVPR 2020: use MVC and let a neural network find the cage vertex positions

Neural cages